2.1 Find Square Roots and Compare Real Numbers

Before You found squares of numbers and compared rational numbers.
Now You will find square roots and compare real numbers.
Why? So you can find side lengths of geometric shapes, as in Ex. 52.

Key Vocabulary
- **square root**
- **radicand**
- **perfect square**
- **irrational number**
- **real numbers**

COMMON CORE

CC.9-12.N.Q.1 Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.*

Recall that the square of 4 is $4^2 = 16$ and the square of -4 is $(-4)^2 = 16$. The numbers 4 and -4 are called the *square roots* of 16. In this lesson, you will find the square roots of nonnegative numbers.

KEY CONCEPT — *For Your Notebook*

Square Root of a Number

Words If $b^2 = a$, then b is a **square root** of a.

Example $3^2 = 9$ and $(-3)^2 = 9$, so 3 and -3 are square roots of 9.

All positive real numbers have two square roots, a positive square root (or *principal* square root) and a negative square root. A square root is written with the radical symbol $\sqrt{\ }$. The number or expression inside a radical symbol is the **radicand.**

radical symbol $\longrightarrow \sqrt{a} \longleftarrow$ radicand

Zero has only one square root, 0. Negative real numbers do not have real square roots because the square of every real number is either positive or 0.

EXAMPLE 1 Find square roots

Evaluate the expression.

a. $\pm\sqrt{36} = \pm 6$ — The positive and negative square roots of 36 are 6 and -6.

b. $\sqrt{49} = 7$ — The positive square root of 49 is 7.

c. $-\sqrt{4} = -2$ — The negative square root of 4 is -2.

READING
The symbol $\pm$ is read as "plus or minus" and refers to both the positive square root and the negative square root.

✓ GUIDED PRACTICE for Example 1

Evaluate the expression.

1. $-\sqrt{9}$ **2.** $\sqrt{25}$ **3.** $\pm\sqrt{64}$ **4.** $-\sqrt{81}$

PERFECT SQUARES The square of an integer is called a **perfect square**. As shown in Example 1, the square root of a perfect square is an integer. As you will see in Example 2, you need to approximate a square root if the radicand is a whole number that is *not* a perfect square.

EXAMPLE 2 Approximate a square root

FURNITURE The top of a folding table is a square whose area is 945 square inches. Approximate the side length of the tabletop to the nearest inch.

Solution

You need to find the side length s of the tabletop such that $s^2 = 945$. This means that s is the positive square root of 945. You can use a table to determine whether 945 is a perfect square.

Number	28	29	30	31	32
Square of number	784	841	900	961	1024

As shown in the table, 945 is *not* a perfect square. The greatest perfect square less than 945 is 900. The least perfect square greater than 945 is 961.

$900 < 945 < 961$ — Write a compound inequality that compares 945 with both 900 and 961.

$\sqrt{900} < \sqrt{945} < \sqrt{961}$ — Take positive square root of each number.

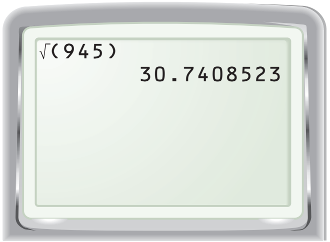
$30 < \sqrt{945} < 31$ — Find square root of each perfect square.

The average of 30 and 31 is 30.5, and $(30.5)^2 = 930.25$. Because $945 > 930.25$, $\sqrt{945}$ is closer to 31 than to 30.

▶ The side length of the tabletop is about 31 inches.

USING A CALCULATOR In Example 2, you can use a calculator to obtain a better approximation of the side length of the tabletop.

2nd √ 945) ENTER



The value shown can be rounded to the nearest hundredth, 30.74, or to the nearest tenth, 30.7. In either case, the length is closer to 31 than to 30.

✓ GUIDED PRACTICE for Example 2

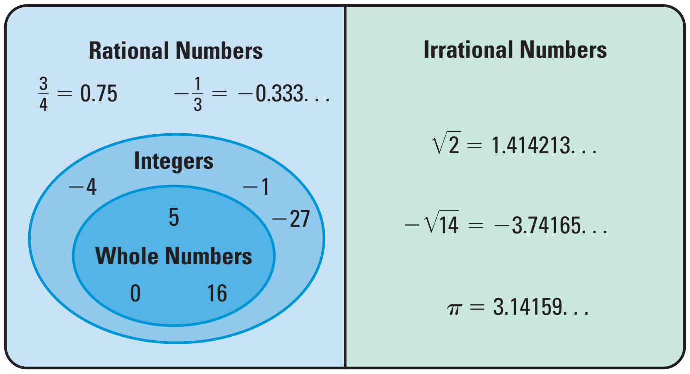
Approximate the square root to the nearest integer.

5. $\sqrt{32}$ **6.** $\sqrt{103}$ **7.** $-\sqrt{48}$ **8.** $-\sqrt{350}$

IRRATIONAL NUMBERS The square root of a whole number that is not a perfect square is an example of an *irrational number*. An **irrational number**, such as $\sqrt{945} = 30.74085...$, is a number that cannot be written as a quotient of two integers. The decimal form of an irrational number neither terminates nor repeats.

REAL NUMBERS The set of **real numbers** is the set of all rational and irrational numbers, as illustrated in the Venn diagram below. Every point on the real number line represents a real number.

REAL NUMBERS



READING
If you exclude 0 from the whole numbers, the resulting set is called the *natural numbers*.

EXAMPLE 3 Classify numbers

Tell whether each of the following numbers is a real number, a rational number, an irrational number, an integer, or a whole number: $\sqrt{24}$, $\sqrt{100}$, $-\sqrt{81}$.

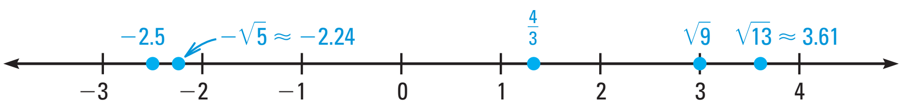
Number	Real number?	Rational number?	Irrational number?	Integer?	Whole number?
$\sqrt{24}$	Yes	No	Yes	No	No
$\sqrt{100}$	Yes	Yes	No	Yes	Yes
$-\sqrt{81}$	Yes	Yes	No	Yes	No

EXAMPLE 4 Graph and order real numbers

Order the numbers from least to greatest: $\frac{4}{3}$, $-\sqrt{5}$, $\sqrt{13}$, -2.5, $\sqrt{9}$.

Solution

Begin by graphing the numbers on a number line.



▸ Read the numbers from left to right: -2.5, $-\sqrt{5}$, $\frac{4}{3}$, $\sqrt{9}$, $\sqrt{13}$.

✓ GUIDED PRACTICE for Examples 3 and 4

9. Tell whether each of the following numbers is a real number, a rational number, an irrational number, an integer, or a whole number: $-\frac{9}{2}$, 5.2, 0, $\sqrt{7}$, 4.1, $-\sqrt{20}$. Then order the numbers from least to greatest.

CONDITIONAL STATEMENTS A conditional statement not in if-then form can be written in that form.

EXAMPLE 5 Rewrite a conditional statement in if-then form

Rewrite the given conditional statement in if-then form. Then tell whether the statement is *true* or *false*. If it is false, give a counterexample.

Solution

a. Given: No fractions are irrational numbers.

If-then form: If a number is a fraction, then it is not an irrational number.

The statement is true.

b. Given: All real numbers are rational numbers.

If-then form: If a number is a real number, then it is a rational number.

The statement is false. For example, $\sqrt{2}$ is a real number but *not* a rational number.

✓ GUIDED PRACTICE for Example 5

Rewrite the conditional statement in if-then form. Then tell whether the statement is *true* or *false*. If it is false, give a counterexample.

10. All square roots of perfect squares are rational numbers.

11. All repeating decimals are irrational numbers.

12. No integers are irrational numbers.

2.1 EXERCISES

HOMEWORK KEY

○ = See **WORKED-OUT SOLUTIONS** Exs. 9, 19, and 47

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 23, 41, 42, 48, and 51

◆ = **MULTIPLE REPRESENTATIONS** Ex. 52

SKILL PRACTICE

1. VOCABULARY Copy and complete: The set of all rational and irrational numbers is called the set of __?__.

2. ★ **WRITING** Without calculating, how can you tell whether the square root of a whole number is rational or irrational?

EXAMPLE 1 for Exs. 3–14

EVALUATING SQUARE ROOTS **Evaluate the expression.**

3. $\sqrt{4}$ **4.** $-\sqrt{49}$ **5.** $-\sqrt{9}$ **6.** $\pm\sqrt{1}$

7. $\sqrt{196}$ **8.** $\pm\sqrt{121}$ **9.** $\pm\sqrt{2500}$ **10.** $-\sqrt{256}$

11. $-\sqrt{225}$ **12.** $\sqrt{361}$ **13.** $\pm\sqrt{169}$ **14.** $-\sqrt{1600}$

EXAMPLE 2
for Exs. 15–22

APPROXIMATING SQUARE ROOTS **Approximate the square root to the nearest integer.**

15. $\sqrt{10}$ **16.** $-\sqrt{18}$ **17.** $-\sqrt{3}$ **18.** $\sqrt{150}$

19. $-\sqrt{86}$ **20.** $\sqrt{40}$ **21.** $\sqrt{200}$ **22.** $-\sqrt{65}$

23. ★ **MULTIPLE CHOICE** Which number is between -30 and -25?

(A) $-\sqrt{1610}$ (B) $-\sqrt{680}$ (C) $-\sqrt{410}$ (D) $-\sqrt{27}$

EXAMPLES 3 AND 4
for Exs. 24–29

CLASSIFYING AND ORDERING REAL NUMBERS **Tell whether each number in the list is a real number, a rational number, an irrational number, an integer, or a whole number. Then order the numbers from least to greatest.**

24. $\sqrt{49}$, 8, $-\sqrt{4}$, -3

25. $-\sqrt{12}$, -3.7, $\sqrt{9}$, 2.9

26. -11.5, $-\sqrt{121}$, -10, $\frac{25}{2}$, $\sqrt{144}$

27. $\sqrt{8}$, $-\frac{2}{5}$, -1, 0.6, $\sqrt{6}$

28. $-\frac{8}{3}$, $-\sqrt{5}$, 2.6, -1.5, $\sqrt{5}$

29. -8.3, $-\sqrt{80}$, $-\frac{17}{2}$, -8.25, $-\sqrt{100}$

EXAMPLE 5
for Exs. 30–33

ANALYZING CONDITIONAL STATEMENTS **Rewrite the conditional statement in if-then form. Then tell whether the statement is *true* or *false*. If it is false, give a counterexample.**

30. All whole numbers are real numbers.

31. All real numbers are irrational numbers.

32. No perfect squares are whole numbers.

33. No irrational numbers are whole numbers.

EVALUATING EXPRESSIONS **Evaluate the expression for the given value of x.**

34. $3 + \sqrt{x}$ when $x = 9$

35. $11 - \sqrt{x}$ when $x = 81$

36. $4 \cdot \sqrt{x}$ when $x = 49$

37. $-7 \cdot \sqrt{x}$ when $x = 36$

38. $-3 \cdot \sqrt{x} - 7$ when $x = 121$

39. $6 \cdot \sqrt{x} + 3$ when $x = 100$

40. **REASONING** Tell whether each of the following sets of numbers has an additive identity, additive inverses, a multiplicative identity, and multiplicative inverses: whole numbers, integers, rational numbers, real numbers. Use a table similar to the one in Example 3 to display your results.

41. ★ **MULTIPLE CHOICE** If $x = 36$, the value of which expression is a perfect square?

(A) $\sqrt{x} + 17$ (B) $87 - \sqrt{x}$ (C) $5 \cdot \sqrt{x}$ (D) $8 \cdot \sqrt{x} + 2$

42. ★ **WRITING** Let $x > 0$. Compare the values of x and $\sqrt{x}$ for $0 < x < 1$ and for $x > 1$. Give examples to justify your thinking.

43. **CHALLENGE** Find the first five perfect squares x such that $2 \cdot \sqrt{x}$ is also a perfect square. *Describe* your method.

44. **CHALLENGE** Let n be any whole number from 1 to 1000. For how many values of n is $\sqrt{n}$ a rational number? *Explain* your reasoning.

PROBLEM SOLVING

EXAMPLE 1 for Exs. 45, 47

45. ART The area of a square painting is 3600 square inches. Find the side length of the painting.

EXAMPLE 2 for Exs. 46, 48

46. SOCCER Some soccer drills are practiced in a square section of a field. If the section of a field for a soccer drill is 1620 square yards, find the side length of the section. Round your answer to the nearest yard.

47. MAZES The table shows the locations and areas of various life-size square mazes. Find the side lengths of the mazes. Then tell whether the side lengths are *rational* or *irrational* numbers.

Location of maze	Area (ft^2)
Dallas, Texas	1225
San Francisco, California	576
Corona, New York	2304
Waterville, Maine	900

Maze at Corona, New York

48. ★ SHORT RESPONSE You plan to use a square section of a park for a small outdoor concert. The section should have an area of 1450 square feet. You have 150 feet of rope to use to surround the section. Do you have enough rope? *Explain* your reasoning.

49. STORAGE CUBES If $b^3 = a$, then b is called the *cube root* of a, written $\sqrt[3]{a}$. For instance, $14^3 = 2744$, so 14 is the cube root of 2744, or $14 = \sqrt[3]{2744}$. Suppose a storage cube is advertised as having a volume of 10 cubic feet. Use a calculator to approximate the edge length of the storage cube to the nearest tenth of a foot.

50. MULTI-STEP PROBLEM The Kelvin temperature scale was invented by Lord Kelvin in the 19th century and is often used for scientific measurements. To convert a temperature from degrees Celsius (°C) to kelvin (K), you add 273 to the temperature in degrees Celsius.

a. Convert 17°C to kelvin.

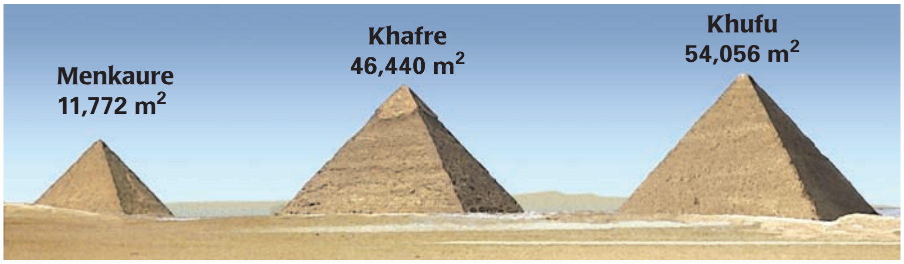
b. The speed s (in meters per second) of sound in air is given by the formula $s = 20.1 \cdot \sqrt{K}$ where K is the temperature in kelvin. Find the speed of sound in air at 17°C. Round your answer to the nearest meter per second.

51. ★ SHORT RESPONSE A homeowner is building a square patio and will cover the patio with square tiles. Each tile has an area of 256 square inches and costs $3.45. The homeowner has $500 to spend on tiles.

a. Calculate How many tiles can the homeowner buy?

b. Explain Find the side length (in feet) of the largest patio that the homeowner can build. *Explain* how you got your answer.

52. **MULTIPLE REPRESENTATIONS** The diagram shows the approximate areas (in square meters) of the square bases for the pyramids of Giza.



a. Making a Table Make a table that gives the following quotients (rounded to the nearest tenth) for each of the 3 pairs of pyramids:

- (area of larger base) ÷ (area of smaller base)
- (side length of larger base) ÷ (side length of smaller base)

For each pair of pyramids, how are the two quotients related?

b. Writing an Equation Write an equation that gives the quotient q of the side lengths as a function of the quotient r of the areas.

53. **CHALLENGE** Write an equation that gives the edge length ℓ of a cube as a function of the surface area A of the cube.

Extension

Use Real and Rational Numbers

GOAL Identify whether sets of rational and irrational numbers are closed under operations.

Key Vocabulary
- **closure**

COMMON CORE

CC.9-12.N.RN.3 Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

RATIONAL AND IRRATIONAL NUMBERS Recall that a *rational number* is a number $\frac{a}{b}$ where a and b are integers with $b \neq 0$. An *irrational number* is any number that cannot be written as a quotient of two integers.

EXAMPLE 1 Sums of rational numbers

Prove that the sum of two rational numbers is rational.

Solution

Let x and y be two rational numbers.

By the definition of rational numbers, x can be written as $\frac{a}{b}$ and y can be written as $\frac{c}{d}$ where a, b, c, and d are integers with $b \neq 0$ and $d \neq 0$.

$x + y = \frac{a}{b} + \frac{c}{d}$ **Add x and y.**

$x + y = \frac{ad + bc}{bd}$ **Rewrite $\frac{a}{b} + \frac{c}{d}$ using a common denominator.**

Because the sum or product of two integers will always be integers, the expressions $ad + bc$ and bd are both integers.

Therefore, the sum $x + y$ is equal to the ratio of two integers. So by definition, this sum is a rational number.

CLOSURE As you saw in Example 1, the sum of two rational numbers is rational. The set of rational numbers has *closure* or is *closed* under multiplication.

KEY CONCEPT *For Your Notebook*

Closure

A set has **closure** or is closed under a given operation if the number that results from performing the operation on any two numbers in the set is also in the set.

Example: The sum of any two rational numbers is a rational number. The set of rationals is closed under addition.

$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ $\qquad$ $\frac{1}{2} + \frac{3}{2} = 2$

Non-example: The quotient of two integers is not necessarily an integer. The set of integers is not closed under division.

$6 \div 2 = 3$ $\qquad$ $-6 \div 5 = -\frac{6}{5}$

EXAMPLE 2 Sum of a rational and an irrational number

Solve the equation $x - 3 = \sqrt{2}$. Is the solution *rational* or *irrational*?

Solution

$x - 3 = \sqrt{2}$ **Write original equation.**

$x - 3 + 3 = \sqrt{2} + 3$ **Add 3 to both sides.**

$x = \sqrt{2} + 3$ **Simplify.**

▶ The solution is *irrational*.

SUMS OF IRRATIONAL NUMBERS Example 2 shows a single case where the sum of a rational number and an irrational number is irrational. To prove that this is always true, you must first assume that such a sum is rational. This results in a contradiction which proves that the assumption must be false.

Let a be rational and b be irrational. Let c be the sum of a and b, and assume that c is rational.

$a + b = c$ **Assume c is rational.**

$b = c - a$ **Subtract a from each side.**

By Example 1, you know that $c - a$ is rational. But a rational number cannot be equal to an irrational number, so this is a contradiction. Therefore the sum of a rational number and an irrational number must be irrational.

PRACTICE

1. Use Example 1 as a model to prove that the product of two rational numbers is rational.

2. Copy and complete: Prove that the product of a nonzero rational number and an irrational number is irrational.

Let x be a nonzero rational number and y be an __?__ number. By definition, $x = \frac{a}{b}$, where a and b are __?__ with $b \neq 0$. Now assume that the product xy is a __?__ number. Therefore xy can be written as the quotient of integers c and d with $d \neq 0$. __?__ **The product xy can be written as $\frac{c}{d}$.** __?__ **Substitute $\frac{a}{b}$ for x.** __?__ **Multiply both sides by __?__.** __?__ **Simplify.** By definition, __?__ is a rational number which means that y must be rational. But y is an irrational number, meaning the assumption that __?__ is rational must be false. Therefore, __?__.

3. Use an indirect proof like the one following Example 2 to prove that the sum of a rational number and an irrational number is irrational.

Investigating Algebra ACTIVITY *Use before Solve One-Step Equations*

Modeling One-Step Equations

MATERIALS • algebra tiles

MATHEMATICAL PRACTICES

Use appropriate tools strategically.

QUESTION **How can you use algebra tiles to solve one-step equations?**

You can model one-step equations using algebra tiles.

1-tile

A 1-tile represents the number 1.

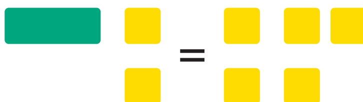
***x*-tile**

An x-tile represents the variable x.

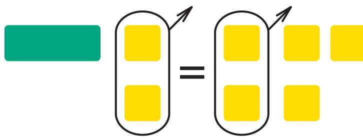
EXPLORE 1 Solve an equation using subtraction

Solve $x + 2 = 5$.

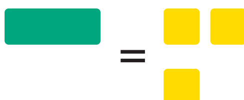
STEP 1 Model $x + 2 = 5$ using algebra tiles.



STEP 2 To find the value of x, isolate the x-tile on one side of the equation. You can do this by removing two 1-tiles from each side.



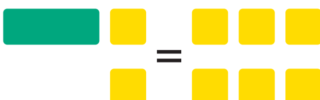
STEP 3 The x-tile is equal to three 1-tiles. So, the solution of $x + 2 = 5$ is 3.



PRACTICE

Write the equation modeled by the algebra tiles.

1.



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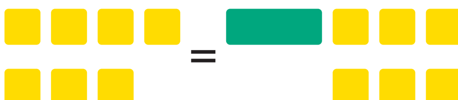

Use algebra tiles to model and solve the equation.

3. $x + 3 = 9$ **4.** $x + 2 = 7$ **5.** $x + 8 = 8$ **6.** $x + 3 = 7$

7. $x + 2 = 12$ **8.** $x + 7 = 12$ **9.** $15 = x + 5$ **10.** $13 = x + 10$

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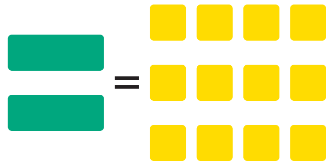
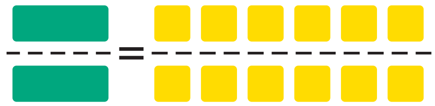
EXPLORE 2 Solve an equation using division

Solve $2x = 12$.

STEP 1 Model $2x = 12$ using algebra tiles.



STEP 2 There are two x-tiles, so divide the x-tiles and 1-tiles into two equal groups.

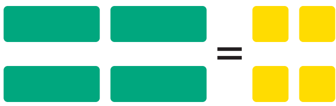


STEP 3 An x-tile is equal to six 1-tiles. So, the solution of $2x = 12$ is 6.

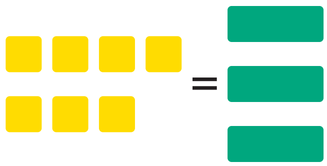
PRACTICE

Write the equation modeled by the algebra tiles.

11.



12.



Use algebra tiles to model and solve the equation.

13. $2x = 10$ **14.** $3x = 12$ **15.** $3x = 18$ **16.** $4x = 16$

17. $6 = 2x$ **18.** $12 = 4x$ **19.** $20 = 5x$ **20.** $21 = 7x$

DRAW CONCLUSIONS Use your observations to complete these exercises

21. An equation and explanation that correspond to each step in Explore 1 are shown below. Copy and complete the equations and explanations.

$x + 2 = 5$ Original equation

$x + 2 - \underline{\ ?\ } = 5 - \underline{\ ?\ }$ Subtract _?_ from each side.

$x = \underline{\ ?\ }$ Simplify. Solution is _?_.

22. Write an equation that corresponds to the algebra tile equation in each step of Explore 2. Based on your results, describe an algebraic method that you can use to solve $12x = 180$. Then use your method to find the solution.