LESSON 25

Weighted Averages and Expected Values

GOAL Find weighted averages, and compute and interpret expected values.

Visitor Ratings
★★★★
★
★★★
★★★★★
★★★
★★★★★
★★
★★★★★
★★★★★
★★★★

Visitors to a Web site give books ratings from 1 to 5 stars. The chart shows the ratings that one book received from 10 visitors to the site.

To find the average rating, you can add the numbers of stars in the ratings and then divide by the number of ratings.

$$\frac{4 + 1 + 3 + 5 + 3 + 5 + 2 + 5 + 5 + 4}{10} = 3.7$$

The average rating is 3.7 stars.

You can also find the average rating by multiplying each rating by the number of times it occurs, adding these products, and then dividing by the total number of ratings.

$$\frac{1(1) + 2(1) + 3(2) + 4(2) + 5(4)}{10} = 3.7$$

Notice that some ratings occur more frequently than others, and these ratings are "weighted" more heavily in the numerator because they are multiplied by greater values. This idea can be generalized to give a definition of a *weighted average*. Note that the weights do not have to be based on frequencies, but can be based on other factors, such as relative importance or value.

Weighted Average

To find the **weighted average** (or *weighted mean*) of a data set, multiply each value by its weight and add the products together, then divide by the sum of the weights.

For a set of data $x_1, x_2, \ldots, x_n$, with nonnegative weights $w_1, w_2, \ldots, w_n$, the weighted average is

$$\frac{x_1w_1 + x_2w_2 + \cdots + x_nw_n}{w_1 + w_2 + \cdots + w_n}.$$

EXAMPLE 1 Finding a Weighted Average

A grade point average is a weighted average that gives greater weight to courses that earn more credits. Hailey's grade points are 4.0 in Chemistry, which is worth 4 credits, 3.5 in English, which is worth 3 credits, and 3.7 in Physics, which is worth 2 credits. What is Hailey's grade point average?

SOLUTION

Multiply each grade by its weight. Add the products and then divide by the sum of the weights.

$$\frac{4.0(4) + 3.5(3) + 3.7(2)}{4 + 3 + 2} = \frac{33.9}{9} = 3.7\overline{6}$$

Hailey's grade point average is approximately 3.77.

Sometimes the weights associated with data values are numbers between 0 and 1 and the sum of the weights is 1. In this case, you can use the following equivalent formula for the weighted average.

Weighted Average (Alternate Definition)

Given a set of data $x_1, x_2, \ldots, x_n$, with nonnegative weights $w_1, w_2, \ldots, w_n$, such that all of the weights are greater than or equal to 0 and less than or equal to 1, and $w_1 + w_2 + \cdots + w_n = 1$, the weighted average is

$$x_2w_1 + x_2w_2 + \cdots + x_nw_n.$$

EXAMPLE 2 Finding a Weighted Average

At a language school, each student is given a score that measures his or her fluency. The fluency score is a weighted average that is determined by rating the student on a scale of 0 to 10 in three categories: grammar, vocabulary, and pronunciation. Grammar counts for 40% of the score, vocabulary counts for 25%, and pronunciation counts for 35%. Thomas gets ratings of 8 for grammar, 6 for vocabulary, and 5 for pronunciation. What is his fluency score?

SOLUTION

The weights are 40% or 0.4, 25% or 0.25, and 35% or 0.35.

The weights are all between 0 and 1, and their sum is 1.

The weighted average is $8(0.4) + 6(0.25) + 5(0.35) = 6.45$.

Thomas's fluency score is 6.45.

CHECK Examples 1 and 2

1. The table shows the value of the tiles in a word game. What is the average value of a tile?

Value of tile	0	1	2	3	4	5	8	10
Number of tiles	2	68	7	8	10	1	2	2

2. A model railroad kit includes straight tracks in three lengths. Sixty percent of the straight tracks are 12 inches long, 30% are 8 inches long, and 10% are 6 inches long. What is the average length of a straight track?

Weighted averages have an important role in probability. To understand this role, it is helpful to first recall that a *random variable* is a variable whose values are determined by the outcome of a probability experiment. The value of the random variable changes as the experiment is repeated.

The **expected value** of a random variable is the long-term average value of the random variable. The following formula shows that the expected value of a discrete random variable is a special case of a weighted average.

Expected Value

For a discrete random variable with possible values $x_1, x_2, \ldots, x_n$ having probabilities $p_1, p_2, \ldots, p_n$, the expected value of the random variable is

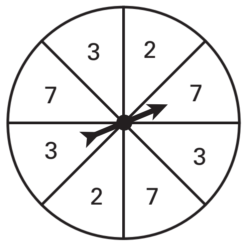
$$x_1p_1 + x_2p_2 + \cdots + x_np_n.$$

EXAMPLE 3 Calculating an Expected Value

Determine the possible values of the discrete random variable. Then find its expected value.

a. You roll a six-sided number cube.

b. You spin the spinner shown, which is divided into equal parts.



SOLUTION

a. The random variable represents the outcome of the roll. The random variable can take on the values 1, 2, 3, 4, 5, and 6.

The probability of each outcome is $\frac{1}{6}$.

The expected value of the roll of the number cube is

$$1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) = 3.5.$$

b. The random variable represents the outcome of the spin. The random variable can take on the values 2, 3, and 7.

$$P(2) = \frac{2}{8} = \frac{1}{4}$$

$$P(3) = \frac{3}{8}$$

$$P(7) = \frac{3}{8}$$

The expected value of the spin is $2\left(\frac{1}{4}\right) + 3\left(\frac{3}{8}\right) + 7\left(\frac{3}{8}\right) = 4.25$.

Notice in Example 3 that the expected value of a random variable need not be one of the possible values of the random variable. For example, when you roll a number cube, the only possible outcomes are 1, 2, 3, 4, 5, and 6. However, if you repeat the experiment many times you will find that the average of the outcomes is close to 3.5.

EXAMPLE 4 Applying and Interpreting Expected Value

On a game show, contestants can spend \$100 of their prize money on an opportunity to roll a giant number cube. If they roll a 6, they get \$300. If they roll a 5, they get \$150. If they roll a 1, 2, 3, or 4, they get nothing. What amount can a contestant expect to win or lose by rolling the cube?

SOLUTION

The random variable represents the net amount won or lost by rolling the cube. Since it costs \$100 to roll the cube, the random variable can take on these values:

$\$300 - \$100 = \mathbf{\$200}$ for rolling a 6
$\$150 - \$100 = \mathbf{\$50}$ for rolling a 5
$\$0 - \$100 = \mathbf{-\$100}$ for rolling a 1, 2, 3, or 4

The probabilities of getting these values are as follows:

$$P(\$200) = P(\text{roll a } 6) = \frac{1}{6}$$

$$P(\$50) = P(\text{roll a } 5) = \frac{1}{6}$$

$$P(-\$100) = P(\text{roll a } 1, 2, 3, \text{ or } 4) = \frac{4}{6} = \frac{2}{3}$$

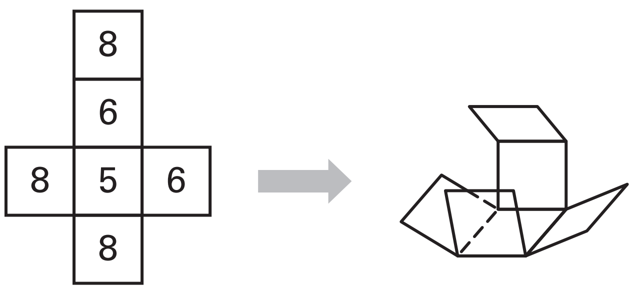
The expected value of the random variable is $\$200\left(\frac{1}{6}\right) + \$50\left(\frac{1}{6}\right) - \$100\left(\frac{2}{3}\right) = -\$25$.

The expected value of $-\$25$ means that, on average, contestants lose \$25 by rolling the giant number cube.

CHECK Examples 3 and 4

3. You have a set of 10 cards that are numbered 1 through 10. You draw a card at random. Find the expected value of the draw.

4. The figure shows a net that can be cut out and folded to make a number cube. You roll the resulting number cube. Find the expected value of the roll.



5. A carnival game costs \$1. Players randomly choose a ball from a bucket containing 100 balls. Five of the balls say "Win a prize!" The value of the prize is \$9. The remaining balls in the bucket are blank. What net amount can a player expect to win or lose by playing the game?

It can sometimes be difficult or impossible to calculate an expected value directly. In this case, you may be able to use a simulation to estimate the expected value.

Activity

Using a Simulation to Find an Expected Value

A cereal manufacturer puts one of six prizes in each box of cereal. Each prize has an equal chance of occurring. How many boxes of cereal would you expect to purchase if you wanted to collect all six prizes? You can use a simulation to help answer this question.

1. Explain how you can use one roll of a number cube to simulate buying one box of cereal.
2. Roll a number cube repeatedly and keep track of the outcomes in a table. Continue to roll the number cube until five of the outcomes have occurred at least once and the sixth outcome has just occurred for the first time.
3. Based on your simulation, how many boxes of cereal did you have to buy in order to collect all six prizes?
4. Repeat the simulation at least five more times. Find the average number of boxes you had to buy in order to collect all six prizes. How does this compare to your answer in Step 3?

EXERCISES

1. During one day at a frozen yogurt shop, 53 customers order 8 ounce servings, 78 customers order 6 ounce servings, and 45 customers order 4 ounce servings. What is the average weight of a serving of frozen yogurt sold on this day?

2. The 30 students in Ms. Chen's class had an average grade of 85 on a standardized test. The 20 students in Mr. Jackson's class had an average grade of 90 on the same test. What is the average test grade of all 50 students?

3. Some colleges use Weighted Average Tuition (WAT) to determine financial aid for students. To calculate WAT, all schools in a college system report their annual tuition and annual attendance. The WAT is the average tuition, weighted by the number of students at each school. What is the WAT for the community college system shown in the table?

Community Colleges

College	Annual tuition	Number of students
Anderson Tech	\$1400	1500
Chavez College	\$2200	3900
Redwood Vocational	\$1800	2500

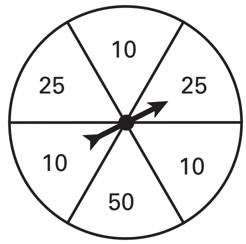
4. Michelle's grade in math class is based on two tests, each worth 25%, and a final exam, worth 50%. Calculate Michelle's math grade if she scored 79% and 82% on the two tests and 85% on the final exam.

5. The Consumer Price Index (CPI) is a weighted average of the change in prices paid by consumers for a representative set of goods and services. Use the table to calculate the CPI from March 2008 to March 2009. Round to the nearest tenth of a percent.

Category	Weight	Percent change (3/08 to 3/09)
Food and Beverages	16%	4.3
Housing	43%	1.4
Apparel	4%	1.4
Transportation	15%	−13.1
Medical Care	6%	2.8
Recreation	6%	1.7
Education and Communication	6%	3.6
Other	4%	5.7

6. A bowl contains 7 slips of paper numbered 4 through 10. You draw a slip of paper at random. Find the expected value of the draw.

7. You spin the spinner shown, which is divided into equal parts. Find the expected value of the spin.



8. A bag contains two red marbles and one green marble. As part of a fundraising event, you pay \$5 to choose a marble without looking. If the marble is green, you get \$10; otherwise, your \$5 goes to a charity.

a. What is the expected amount that you win or lose by playing one time?

b. Suppose 100 people play the game during the event. How much money can the organizers expect to raise for the charity?

9. On a standardized test, each multiple choice question has five choices. A student receives 1 point for a correct answer and loses $\frac{1}{4}$ point for an incorrect answer. Suppose a student randomly guesses the answer to a question.

a. What is the student's expected score on the question?

b. Suppose the student can determine that one of the answer choices is incorrect. Is it worthwhile to make a random guess among the other choices? Justify your decision.

10. Amy and Kendra play a game with the following rules. They start with the same number of chips and take turns rolling two number cubes. If the sum of the numbers on the cubes is 6 or less, Amy gives Kendra 7 chips. If the sum of the numbers on the cubes is 7 or more, Amy takes 5 chips from Kendra. The girls continue play until the winner has all of the chips. Is the game fair? That is, does each player have an equal chance of winning? Why or why not?

11. During a supermarket's first week in business, 100 random customers received a \$5 coupon and 500 random customers received a \$10 coupon. The second week, 350 random customers received a \$6 coupon and 300 random customers received a \$10 coupon. An advertisement claimed that the coupon giveaway increased during the second week. Use weighted averages or expected values to explain whether you agree or disagree with this claim.