

Binomial Distributions

GOAL Use Pascal's triangle to find combinations and generate binomial distributions.

Recall that a *combination* is an arrangement of a group of objects where the order is not important. The number of combinations of r objects taken from a group of n distinct objects is denoted ${}_n C_r$ and can be found using the formula

$${}_n C_r = \frac{n!}{(n-r)! \cdot r!}$$

You can also use *Pascal's triangle* to find combinations. **Pascal's triangle** consists of rows of numbers in which the numbers on the ends of each row are 1 and every other number is the sum of the two numbers diagonally above it. The figure shows the first few rows of Pascal's triangle.

Row 0 →					1				
Row 1 →				1		1			
Row 2 →			1		2		1		
Row 3 →		1		3		3		1	
Row 4 →	1		4		6		4		1

The numbers in the n th row of Pascal's triangle are the combinations of n objects taken 0 objects at a time, 1 object at a time, 2 objects at a time, and so on, up to n objects at a time. The table shows this relationship for the first few rows of Pascal's triangle.

	Pascal's Triangle	Combinations
Row 0	1	${}_0 C_0$
Row 1	1 1	${}_1 C_0$ ${}_1 C_1$
Row 2	1 2 1	${}_2 C_0$ ${}_2 C_1$ ${}_2 C_2$
Row 3	1 3 3 1	${}_3 C_0$ ${}_3 C_1$ ${}_3 C_2$ ${}_3 C_3$
Row 4	1 4 6 4 1	${}_4 C_0$ ${}_4 C_1$ ${}_4 C_2$ ${}_4 C_3$ ${}_4 C_4$

EXAMPLE 1 Using Pascal's Triangle to Find Combinations

Use Pascal's triangle to find ${}_5 C_3$.

SOLUTION

First extend Pascal's triangle to row 5.

Row 0 →					1						
Row 1 →				1		1					
Row 2 →			1		2		1				
Row 3 →		1		3		3		1			
Row 4 →		1		4		6		4		1	
Row 5 →	1		5		10		10		5		1

Then identify ${}_5 C_3$ in row 5.

$${}_5 C_3 = 10$$

**CHECK Example 1**

1. Use Pascal's triangle to find ${}_6C_4$.

Binomial Experiment

A **binomial experiment** is an experiment that meets the following criteria.

- There are n independent trials.
- Each trial has only two possible outcomes, success or failure.
- The probability of success p is the same for each trial, and the probability of failure q is the same for each trial. Since there are only two outcomes in each trial, $p + q = 1$ and $q = 1 - p$.

Suppose you roll a number cube 20 times and are interested in the number of times you roll a 6. This is a binomial experiment in which there are 20 trials, the probability p of success (rolling a 6) is $\frac{1}{6}$ and the probability q of failure (rolling any other number) is $\frac{5}{6}$.

The following formula shows how binomial experiments are related to combinations.

Binomial Probability

If a binomial experiment has n trials in which p is the probability of success and q is the probability of failure in any given trial, then the **binomial probability** that there will be exactly r successes is given by

$$P(r) = {}_n C_r p^r q^{n-r}.$$

EXAMPLE 2 Finding Binomial Probabilities

You flip a coin 5 times. What is the probability of getting exactly 3 heads?

SOLUTION

For this experiment, the number n of trials is 5. The probability p of success (the coin lands heads up) is $\frac{1}{2}$, or 0.5. The probability q of failure (the coin lands tails up) is also $\frac{1}{2}$, or 0.5.

$$P(r) = {}_n C_r p^r q^{n-r}$$

$$P(3) = {}_5 C_3 (0.5)^3 (0.5)^2 \quad \text{Substitute 5 for } n, 3 \text{ for } r, \text{ and } 0.5 \text{ for } p \text{ and } q.$$

$$= 10(0.125)(0.25)$$

$$= 0.3125$$

The probability of getting exactly 3 heads is about 31%.

EXAMPLE 3 Finding Binomial Probabilities

You flip a coin 5 times. What is the probability of getting at least 3 heads?

SOLUTION

The probability of getting at least 3 heads is the probability of getting exactly 3 heads, 4 heads, or 5 heads. Add the individual probabilities.

$$\begin{aligned} P(\geq 3) &= P(3) + P(4) + P(5) \\ &= 0.3125 + {}_5C_4(0.5)^4(0.5)^{5-4} + {}_5C_5(0.5)^5(0.5)^{5-5} \\ &= 0.3125 + 0.15625 + 0.03125 = 0.5 \end{aligned}$$

The probability of getting at least 3 heads is 0.5 or 50%.

A **binomial distribution** is a probability distribution that shows the probabilities of the outcomes a binomial experiment. You can display a binomial distribution in a table or histogram. A graphing calculator makes it easy to find the required probabilities.

EXAMPLE 4 Generating a Binomial Distribution

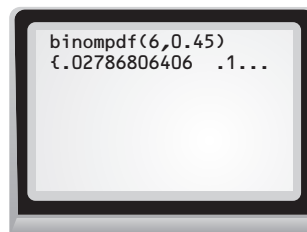
A cereal manufacturer randomly puts a prize in 45% of all cereal boxes. You buy 6 boxes of cereal. Make a table and a histogram for this binomial distribution.

SOLUTION

For this experiment, the number of trials n is 6. The probability p of success (buying a box with a prize) is 0.45.

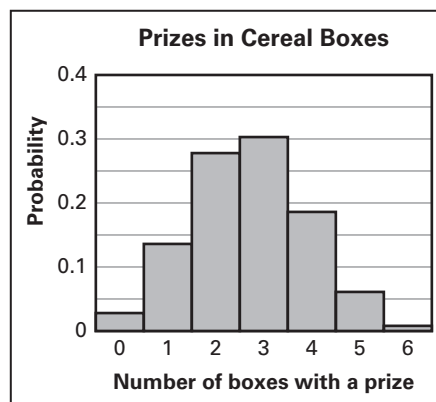
Rather than calculating $P(0), P(1), \dots, P(6)$ individually, use your calculator to find these values.

Use the binomial distribution function, **binompdf**(n, p), where n is the number of trials and p is the probability of success. The calculator returns a horizontal list of the binomial probabilities.



Record the probabilities in a table and use the table to make a histogram.

Number of boxes with a prize	Probability
0	≈ 0.028
1	≈ 0.136
2	≈ 0.278
3	≈ 0.303
4	≈ 0.186
5	≈ 0.061
6	≈ 0.008



**CHECK** Examples 2, 3, and 4

In Exercises 2–4, use the following information.

You spin a spinner 4 times. The spinner has one red, one blue, and one green sector, each of equal area. You win a prize if the spinner lands on the red sector.

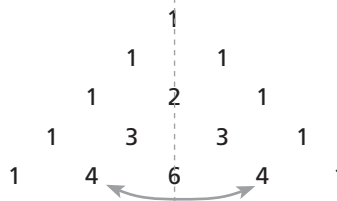
- What is the probability that the spinner lands on red exactly 2 times?
- What is the probability that the spinner lands on red at least 2 times?
- Make a table and a histogram for this binomial distribution.

EXERCISES

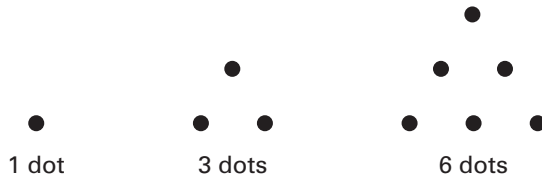
Use Pascal's triangle to find the value of the expression.

- | | | | |
|--------------|------------------|---------------|------------------|
| 1. ${}_3C_2$ | 2. ${}_4C_1$ | 3. ${}_6C_3$ | 4. ${}_6C_4$ |
| 5. ${}_8C_5$ | 6. ${}_7C_3$ | 7. ${}_9C_4$ | 8. ${}_8C_3$ |
| 9. ${}_7C_7$ | 10. ${}_{10}C_4$ | 11. ${}_9C_1$ | 12. ${}_{10}C_8$ |

13. As shown in the figure, Pascal's triangle is symmetric about a vertical line. Suppose you know the value of ${}_{12}C_2$. Explain how you can use symmetry to find ${}_{12}C_{10}$.



14. For any whole number n , the value of ${}_nC_n$ is 1.
- Use Pascal's triangle to explain why this is true.
 - Use the formula for combinations to show why this is true.
 - Explain why it makes sense that ${}_nC_n = 1$.
15. The *triangular numbers* are obtained by arranging dots to form equilateral triangles, as shown. The triangular numbers are the numbers of dots in the triangles. The figure shows that the first three triangular numbers are 1, 3, and 6.



- Explain how the triangular numbers are related to Pascal's triangle.
 - Use Pascal's triangle to find the next three triangular numbers.
16. The sums of the numbers in the rows of Pascal's triangle form an interesting pattern.
- Find the sum of the numbers in each row of Pascal's triangle for the first five rows.
 - Make a conjecture about the sum of the numbers in the n th row of Pascal's triangle.

In Exercises 17–19, you roll a number cube 4 times.

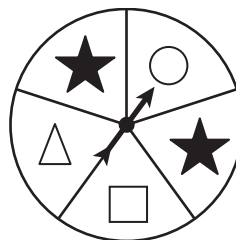
17. What is the probability that you roll a 1 exactly 2 times?
18. What is the probability that you roll a 1 at least 2 times?
19. What is the probability that you do not roll a 1 on any of the rolls?

In Exercises 20–22, use the following information.

In the United States, 81% of the population lives in an urban area (that is, in a city or in a suburb).

20. Suppose you choose 3 U.S. residents at random. What is the probability that exactly 2 of them live in an urban area?
21. Suppose you choose 10 U.S. residents at random. What is the probability that at least 7 of them live in an urban area?
22. Suppose you choose 20 U.S. residents at random. What is the probability that exactly 15 of them live in an urban area? (*Hint:* Use the binomial distribution function of your calculator, $\text{binompdf}(n,p,r)$, where n is the number of trials, p is the probability of success, and r is the number of successes.)

23. Players on a game show get 3 chances to spin the spinner shown, which is divided into equal parts.
 - a. Make a table for the binomial distribution where the random variable is the number of stars a player gets in three spins.
 - b. Find the sum of the probabilities in the table. Explain why this sum makes sense.



24. A history quiz consists of 5 multiple choice questions. Each question has 4 answer choices, A, B, C, and D. Each answer choice is equally likely to be correct and you make a random guess to answer each question.
 - a. Make a table for the binomial distribution where the random variable is the number of questions you answer correctly.
 - b. Make a histogram for the distribution.
 - c. What is the most likely number of questions you will answer correctly?
 - d. How likely is it that you will answer all five questions correctly? Explain.

In Exercises 25 and 26, use the following information.

The table shows the percent of flights by U.S. airlines that were on time, delayed, or cancelled in 2008.

Airlines' On-Time Performance, 2008	
On time	72.6%
Delayed	24.5%
Cancelled	2.7%

25. Your family plans a vacation that involves taking 4 flights on U.S. airlines. What is the probability that at least 3 of the flights will be on time?
26. Which is more likely: a trip involving 6 flights in which at least 2 are delayed or a trip involving 3 flights in which all of them are on time? Justify your decision.