Name ______________________________ Date ________________

# CHAPTER 3 Integer Solutions of Linear Equations

Typically, the variables in a linear equation range over all real numbers. The graph of a linear equation in two variables is a continuous line.

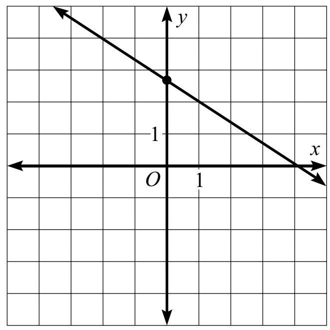
## EXAMPLE 1 Find solutions of linear equations

**a.** Graph the equation $4x + 6y = 16$.

**b.** List some ordered pair solutions, including solutions where $x$ and/or $y$ have non-integer values.

**Solution:**

**a.** We simplify the equation and rewrite it in the form $y = -\frac{2}{3}x + \frac{8}{3}$ to find that the line has a slope of $-\frac{2}{3}$ and a $y$-intercept of $\frac{8}{3}$.




**b.** Substituting arbitrarily selected values of $x$ into $y = -\frac{2}{3}x + \frac{8}{3}$, we generate a list of ordered pairs.

| $y$ | $x$ |
|---|---|
| $-2$ | $4$ |
| $0$ | $\frac{8}{3}$ |
| $\frac{1}{2}$ | $\frac{7}{3}$ |
| $4$ | $0$ |
| $5$ | $-\frac{2}{3}$ |

We can confirm by inspection that the ordered pairs lie on the line in part a. ■

Although it is common for $x$ and $y$ to range over the real numbers, there are many situations where the solutions to an equation can only involve integer values. For instance, an equation might relate numbers of male and female students, or it might relate two elements in a chemical formula. Integer-valued variables such as this are the most common type of *discrete* variables.

## EXAMPLE 2 Find integer solutions of linear equations

Suppose that for the equation $4x + 6y = 16$, we require $x$ and $y$ to be integers. Identify two solutions from Example 1 and find at least one more solution.

**Solution:**

From the table in Example 1, $x = -2$, $y = 4$ and $x = 4$, $y = 0$ are both solutions. Since the solutions lie along a line with a slope of $-\frac{2}{3}$, it should be possible to find more solutions by starting with a known solution and counting up and over the appropriate number of units. Starting at $(-2, 4)$ and counting right 3 and down 2, we find the point

(1, 2) on the line, which means that $x = 1, y = 2$ is a third integer solution. (We could equally well have counted left 3 and up 2 from point (4, 0).) Additional applications of the same process yield $x = -5, y = 6$ and $x = 7, y = -2$ as solutions. ■

Equations with integer-only solutions are called *Diophantine equations*, after the mathematician Diophantus of Alexandria (200–284 AD). The constants in such equations are normally integers, as well. So in a linear Diophantine equation of the form $ax + by = c$, the constants $a$, $b$ and $c$ are integers, and integer values for $x$ and $y$ are sought.

We found the integer solutions in Example 2 by inspecting the graph. There are, however, ways to find the solutions algebraically. The standard method is surprisingly complicated; it involves something called the Euclidean algorithm. But there is also a simpler method, which uses carefully controlled trial and error. This method is based on the fact that any solutions will occur at fixed intervals along the graph of the continuous (non-discrete) version of the equation.

**KEY CONCEPT**

## Finding Integer Solutions for an Equation of the Form $ax + by = c$

1. Find the greatest common factor (GCF) of $a$ and $b$, and divide this number into $|b|$. Call the result $d$.
2. Pick any $d$ consecutive integer values for $x$, and try them until you find one that produces an integer value for $y$. It helps to rewrite the equation in the form $y = -\frac{a}{b}x + \frac{c}{b}$.
3. If Step 2 produces no solutions, the equation has no integer solutions anywhere. If Step 2 produces a solution, then the equation has a series of solutions for which the $x$-values are spaced $d$ units apart. (The spacing of the $y$-values will be the quotient of $|a|$ and the GCF of $a$ and $b$.)

**EXAMPLE 3**

## Find integer solutions of an equation of the for $ax + by = c$

Use controlled trial and error to find integer solutions for the linear equation $4x + 6y = 16$.

**Solution:**

1. The GCF of 4 and 6 is 2, and $|6| \div 2 = 3$.
2. Write the equation as $y = -\frac{2}{3}x + \frac{8}{3}$ and try any 3 consecutive integer values of $x$. We arbitrarily select $x = 0$ as a convenient starting value.

| $x$ | $y$ | |
|---|---|---|
| 0 | $\frac{8}{3}$ | $y$ is not an integer |
| 1 | 2 | $y$ is an integer - STOP |
| 2 | | |

**3.** The ordered pair (1, 2) is a solution. Other solutions are found by changing the value of $x$ up or down in increments of 3:

| $x$ | $y$ |
|---|---|
| −5 | 6 |
| −2 | 4 |
| **1** | **2** |
| 4 | 0 |
| 7 | −2 |

Notice that the quotient of $a = 4$ and 2 (the GFC of 4 and 6) is 2, and the $y$-values occur at 2-unit intervals. The complete range of solutions can be represented as $x = 1 + 3n$, $y = 2 - 2n$, for $n = \ldots, -2, -1, 0, 1, 2, \ldots$. ■

The trial and error method is not very efficient when one has to try many values. If $|a| < |b|$, it may be faster to divide the greatest common factor of $a$ and $b$ into $|a|$ and then try successive values of $y$.

## Practice

**Use controlled trial and error to find the integer solutions, if any, for each linear equation. Write them in terms of *n*, with *n* = …, −2, −1, 0, 1, 2, ….**

**1.** $5x + 2y = 1$

**2.** $4x - 3y = -7$

**3.** $2x + 7y = -10$

**4.** $4x - y = 7$

**5.** $-x + 3y = -9$

**6.** $x + 8y = 0$

**7.** $-6x + 9y = 11$

**8.** $-3x - 9y = 15$

**9.** $-12x + 4y = 6$

**10.** $-11x + 6y = 17$

**11.** $5x + 7y = -9$

**12.** $30x - 35y = 19$

**13.** Several ranchhands are in a corral, working with one or more horses. Counting both human and horse legs, there are 22 legs in the corral. How many cowboys and how many horses might there be? List all the possibilities.

**14.** One molecule of a certain chemical compound is known to have a mass of 160 atomic mass units (amu's). The molecule is believed to consist of carbon (12 amu per atom) and oxygen (16 amu per atom). Based on this information alone, what are the possible formulas for the compound, in the form $C_x O_y$, where $x$ is the number of carbon atoms and $y$ is the number of oxygen atoms?

**15.** **Explain** Suppose that the mass of the molecule in Exercise 14 were remeasured and found to be 162 atomic mass units. Explain why, with this mass, the compound would have to contain something else besides carbon and oxygen.