LESSON 3

Errors in Measurement

GOAL Calculate absolute and relative error, and determine how measurement errors affect calculated measurements.

If you measure the height of a doorway and your measurement is $\frac{1}{2}$ foot from the actual height, your measurement is not very accurate. If you measure the height of a skyscraper and your measurement is $\frac{1}{2}$ foot from the actual height, your measurement is quite impressive. *Absolute error* and *relative error* provide two ways to describe the accuracy of measurements.

Absolute Error

Absolute error is the absolute value of the difference between a measured value and the actual or accepted value.

$$\text{Absolute error} = |\text{measured value} + \text{actual value}|$$

EXAMPLE 1 Calculating Absolute Error

The actual mass of an apple is 0.175 kg. A student uses a balance scale to determine that the mass of the apple is 168 g. Find the absolute error.

SOLUTION

The measured value is 168 g or 0.168 kg. The actual value is 0.175 kg.

$$\text{Absolute error} = |\text{measured value} - \text{actual value}|$$

$$= |0.168 - 0.175| = |-0.007| = 0.007 \text{ kg}$$

The absolute error is 0.007 kg or 7 g.

Unlike absolute error, relative error takes the size of the measurement into account.

Relative Error

Relative error is the absolute error of a measurement divided by the actual value. Relative error is usually expressed as a percent.

$$\text{Relative error} = \frac{\text{absolute error}}{\text{actual value}}, \text{ written as a percent}$$

In the situation presented at the beginning of the lesson, the absolute error in measuring the height of the doorway and in measuring the height of the skyscraper is $\frac{1}{2}$ foot. However, the relative error in measuring the doorway is much greater than the relative error in measuring the skyscraper.

EXAMPLE 2 Calculating Relative Error

The actual mass of an apple is 0.175 kg. A student uses a balance scale to determine that the mass of the apple is 168 g. Find the relative error.

SOLUTION

The absolute error is 0.007 kg, as shown in Example 1.

Relative error $= \dfrac{\text{absolute error}}{\text{actual value}} = \dfrac{0.007}{0.175} = 0.04 = 4\%$

The relative error is 4%.

CHECK Examples 1 and 2

In 1852, the elevation of Mount Everest was measured at 29,002 feet. Today, the accepted elevation for Mount Everest is 29,035 feet.

1. Find the absolute error.

2. Find the relative error.

A *direct measurement* is a measurement that is obtained from using a measurement tool, such as a ruler or scale. A *calculated measurement* is a measurement, such as area or volume, that is calculated from direct measurements. Errors or uncertainties in direct measurements can lead to wide variations in calculated measurements.

EXAMPLE 3 Analyzing Calculated Measurements

A student finds that the length of a concrete wall to the nearest meter is 8 m. Another student finds that the height of the wall to the nearest 0.1 meter is 2.4 m. Find the minimum possible area and maximum possible area of the wall.

SOLUTION

The length of the wall is 8 m $\pm$ 0.5 m. The height of the wall is 2.4 m $\pm$ 0.05 m.

To find the minimum possible area, multiply the minimum length and minimum height:

$$A = 7.5 \text{ m} \times 2.35 \text{ m} = 17.625 \text{ m}^2$$

To find the maximum possible area, multiply the maximum length and maximum height:

$$A = 8.5 \text{ m} \times 2.45 \text{ m} = 20.825 \text{ m}^2$$

The minimum possible area is 17.625 m^2. The maximum possible area is 20.825 m^2.

CHECK Example 3

3. To the nearest 0.1 m, the concrete wall in Example 3 is 0.2 m thick. Find the minimum possible volume and maximum possible volume of the wall.

Given a measurement m_1 with relative error r_1 and a measurement m_2 with relative error r_2, the product of the measurements $m_1 m_2$ has relative error $R = \sqrt{r_1^{\,2} + r_2^{\,2}}$.

EXAMPLE 4 Finding the Relative Error of a Calculated Measurement

The actual length of a rectangle is 8.6 cm and the measured length is 8.4 cm. The actual width of the rectangle is 5.8 cm and the measured width is 5.9 cm. Find the relative error when these measurements are used to calculate the area of the rectangle.

SOLUTION

The relative error in measuring the length is $r_1 = \dfrac{|8.4 - 8.6|}{8.6} = \dfrac{0.2}{8.6} \approx 0.023.$

The relative error in measuring the width is $r_2 = \dfrac{|5.9 - 5.8|}{5.8} = \dfrac{0.1}{5.8} \approx 0.017.$

The relative error in the area is $R = \sqrt{r_1^{\,2} + r_2^{\,2}}$

$$= \sqrt{\left(\frac{0.2}{8.6}\right)^2 + \left(\frac{0.1}{5.8}\right)^2 \approx 0.029}$$

The relative error in calculating the area is about 2.9%.

CHECK Example 4

4. The actual length of the side of a square tile is 6.1 in. and the measured length is 6 in. What is the relative error when this measurement is used to calculate the area of the tile?

In Example 4, notice how R is greater than both r_1 and r_2. This shows how errors in direct measurements are compounded or "propagated" in calculated measurements. The following activity shows how rounding can also lead to errors in calculated measurements.

Activity

Comparing Rounding Errors

The area A of a circle with radius r is given by $A = \pi r^2$.

1 Use the following methods to find the area of a circle with radius 54 cm.

Method 1 Square the radius and round the result to 2 significant digits. Then multiply by 3.14 and round the result to 2 significant digits.

Method 2 Square the radius and multiply by π using the π key on your calculator. Round the result to 2 significant digits.

2 Compare your results. Which calculation gives a more accurate result?

The activity shows that errors can be introduced by rounding measurements at an intermediate stage of a calculation. In general, you get more accurate results when you round only in the last step of a calculation.

EXERCISES

Find the absolute error and the relative error. If necessary, round the relative error to the nearest tenth of a percent.

1. Actual value: 42 m
Measured value: 40 m

2. Actual value: 8.2 cm
Measured value: 7.9 cm

3. Actual value: 23.5 in.
Measured value: 2 ft

4. Actual value: 1525 mg
Measured value: 1.5 g

5. Actual value: 125 sec
Measured value: 2 min

6. The Greek mathematician Eratosthenes (ca 276 BCE – 196 BCE) used shadows and basic geometry to determine that the circumference of Earth is about 25,000 mi. The actual circumference of Earth is 24,902 mi. What was Eratosthenes's absolute error? What was his relative error?

7. Ryan measured a stick that is 16.5 cm long and reported the length as 16 cm. Nadine measured a stick that is 30.2 cm long and reported the length as 31 cm. Who made a more accurate measurement? Justify your choice.

8. A rock has a mass of 2.4 kg. A student measured the mass of the rock with a relative error of 5%.

a. What is the absolute error of the student's measurement?

b. What are the possible values that the student could have found for the rock's mass?

9. An interior designer estimates that the length and width of a rectangular room in a restaurant are 60 feet and 50 feet. The actual length and width are 63 feet and 52 feet.

a. Find the estimated perimeter and the actual perimeter of the room.

b. Find the absolute error and the relative error in the estimated perimeter. Round the relative error to the nearest percent.

c. A wallpaper border comes in rolls that are 15 feet long. The interior designer claims that the relative error is so small that the estimated perimeter can be used to determine how many rolls to buy. Do you agree? Explain why or why not.

10. Is it possible for the relative error of a measurement to be greater than 100%? If not, explain why not. If so, give an example of a measurement with a relative error greater than 100%.

11. The length of a rectangular mouse pad to the nearest centimeter is 21 cm. The mouse pad's width to the nearest 0.1 cm is 12.8 cm. Find the minimum and maximum possible areas of the mouse pad.

12. The area of a parallelogram is the product of its base and its height. The table gives the actual and measured dimensions of a parallelogram.

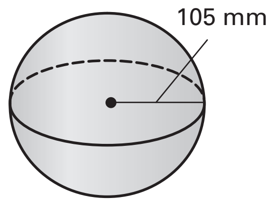
Parallelogram Dimensions		
	Actual	**Measured**
Base	63 mm	65 mm
Height	45 mm	42 mm

a. Find the relative error in the measurements of the base and height.

b. Find the relative error when these measurements are used to calculate the area.

c. How does the relative error in part (b) compare to the relative errors in part (a)?

13. The volume V of a sphere with radius r is given by $V = \frac{4}{3}\pi r^3$.



a. Find the volume of a sphere with radius 105 mm by first finding 105^3, rounding to 3 significant digits, and then using 3.14 for π. Round the resulting volume to 3 significant digits.

b. How does the volume you found in part (a) compare to the volume that you get when you find 105^3 without rounding, use the π key on your calculator, and round to 3 significant digits only at the end of your calculations?

14. A student writes a computer program in which all input and output values are truncated by dropping all the digits after the tenths place. For example, 4.295 is truncated to 4.2. The student uses the program to calculate 3.59×6.15. What value does the program output? What are the absolute and relative errors for this calculation?

15. It is possible to analyze measurement errors when the actual or accepted value is not known. In this case, the *greatest possible error* of a measurement is defined to be half the unit of measure. For example, given a measurement of 4.3 cm, the unit of measure is 0.1 cm and the greatest possible error is half of 0.1 cm or 0.05 cm. The *relative error* is the ratio of the greatest possible error and the measurement. For example, the relative error in the measurement 4.3 cm is $\frac{0.05}{4.3} \approx 1.2\%$.

a. A computer screen is 1.3 ft long. A movie screen is 62 ft long. Find the relative error of each measurement.

b. *Accuracy* may be defined as follows. Given two measurements, the measurement with the smaller relative error is the more accurate measurement. Using this definition of accuracy, tell which measurement in part (a) is more accurate.