6.1 Use Combinations and the Binomial Theorem

Before You used the counting principle and permutations.

Now You will use combinations and the binomial theorem.

Why? So you can find ways to form a set, as in Example 2.

Key Vocabulary
- **combination**
- **Pascal's triangle**
- **binomial theorem**

COMMON CORE

CC.9-12.A.APR.5 (+) Know and apply the Binomial Theorem for the expansion of $(x + y)^n$ in powers of x and y for a positive integer n, where x and y are any numbers, with coefficients determined for example by Pascal's Triangle.

You have learned that order is important for some counting problems. For other counting problems, order is not important. For instance, if you purchase a package of trading cards, the order of the cards inside the package is not important. A **combination** is a selection of r objects from a group of n objects where the order is not important.

KEY CONCEPT — *For Your Notebook*

Combinations of *n* Objects Taken *r* at a Time

The number of combinations of r objects taken from a group of n distinct objects is denoted by ${}_nC_r$ and is given by this formula:

$${}_nC_r = \frac{n!}{(n-r)! \cdot r!}$$

EXAMPLE 1 Find combinations

CARDS A standard deck of 52 playing cards has 4 suits with 13 different cards in each suit.

a. If the order in which the cards are dealt is not important, how many different 5-card hands are possible?

b. In how many 5-card hands are all 5 cards of the same color?

Standard 52-Card Deck

K ♠	K ♥	K ♦	K ♣
Q ♠	Q ♥	Q ♦	Q ♣
J ♠	J ♥	J ♦	J ♣
10 ♠	10 ♥	10 ♦	10 ♣
9 ♠	9 ♥	9 ♦	9 ♣
8 ♠	8 ♥	8 ♦	8 ♣
7 ♠	7 ♥	7 ♦	7 ♣
6 ♠	6 ♥	6 ♦	6 ♣
5 ♠	5 ♥	5 ♦	5 ♣
4 ♠	4 ♥	4 ♦	4 ♣
3 ♠	3 ♥	3 ♦	3 ♣
2 ♠	2 ♥	2 ♦	2 ♣
A ♠	A ♥	A ♦	A ♣

Solution

a. The number of ways to choose **5** cards from a deck of **52** cards is:

$${}_{52}C_5 = \frac{52!}{47! \cdot 5!} = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot \cancel{47!}}{\cancel{47!} \cdot 5!} = 2{,}598{,}960$$

b. For all 5 cards to be the same color, you need to choose **1** of the **2** colors and then **5** of the **26** cards in that color. So, the number of possible hands is:

$${}_2C_1 \cdot {}_{26}C_5 = \frac{2!}{1! \cdot 1!} \cdot \frac{26!}{21! \cdot 5!} = \frac{2}{1 \cdot 1} \cdot \frac{26 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot \cancel{21!}}{\cancel{21!} \cdot 5!} = 131{,}560$$

MULTIPLE EVENTS When finding the number of ways both an event A *and* an event B can occur, you need to multiply, as in part (b) of Example 1. When finding the number of ways that event A *or* event B can occur, you add instead.

EXAMPLE 2 Decide to multiply or add combinations

THEATER William Shakespeare wrote 38 plays that can be divided into three genres. Of the 38 plays, 18 are comedies, 10 are histories, and 10 are tragedies.

a. How many different sets of *exactly* 2 comedies and 1 tragedy can you read?

b. How many different sets of *at most* 3 plays can you read?

AVOID ERRORS
When finding the number of ways to select *at most* n objects, be sure to include the possibility of selecting 0 objects.

Solution

a. You can choose 2 of the 18 comedies and 1 of the 10 tragedies. So, the number of possible sets of plays is:

$$_{18}C_2 \cdot {}_{10}C_1 = \frac{18!}{16! \cdot 2!} \cdot \frac{10!}{9! \cdot 1!} = \frac{18 \cdot 17 \cdot \cancel{16!}}{\cancel{16!} \cdot 2 \cdot 1} \cdot \frac{10 \cdot \cancel{9!}}{\cancel{9!} \cdot 1} = 153 \cdot 10 = 1530$$

b. You can read 0, 1, 2, or 3 plays. Because there are 38 plays that can be chosen, the number of possible sets of plays is:

$$_{38}C_0 + {}_{38}C_1 + {}_{38}C_2 + {}_{38}C_3 = 1 + 38 + 703 + 8436 = 9178$$

SUBTRACTING POSSIBILITIES Counting problems that involve phrases like "at least" or "at most" are sometimes easier to solve by subtracting possibilities you do not want from the total number of possibilities.

EXAMPLE 3 Solve a multi-step problem

BASKETBALL During the school year, the girl's basketball team is scheduled to play 12 home games. You want to attend *at least* 3 of the games. How many different combinations of games can you attend?

Solution

Of the 12 home games, you want to attend 3 games, or 4 games, or 5 games, and so on. So, the number of combinations of games you can attend is:

$$_{12}C_3 + {}_{12}C_4 + {}_{12}C_5 + \cdots + {}_{12}C_{12}$$

Instead of adding these combinations, use the following reasoning. For each of the 12 games, you can choose to attend or not attend the game, so there are 2^{12} total combinations. If you attend at least 3 games, you do not attend only a total of 0, 1, or 2 games. So, the number of ways you can attend at least 3 games is:

$$2^{12} - ({}_{12}C_0 + {}_{12}C_1 + {}_{12}C_2) = 4096 - (1 + 12 + 66) = 4017$$

✓ GUIDED PRACTICE for Examples 1, 2, and 3

Find the number of combinations.

1. $_8C_3$ **2.** $_{10}C_6$ **3.** $_7C_2$ **4.** $_{14}C_5$

5. WHAT IF? In Example 2, how many different sets of *exactly* 3 tragedies and 2 histories can you read?

PASCAL'S TRIANGLE If you arrange the values of ${}_nC_r$ in a triangular pattern in which each row corresponds to a value of n, you get what is called **Pascal's triangle**. Pascal's triangle is named after the French mathematician Blaise Pascal (1623–1662).

KEY CONCEPT — *For Your Notebook*

Pascal's Triangle

Pascal's triangle is shown below with its entries represented by combinations and with its entries represented by numbers. The first and last numbers in each row are 1. Every number other than 1 is the sum of the closest two numbers in the row directly above it.

	Pascal's triangle as combinations	Pascal's triangle as numbers
$n = 0$ (0th row)	${}_0C_0$	1
$n = 1$ (1st row)	${}_1C_0$ ${}_1C_1$	1 1
$n = 2$ (2nd row)	${}_2C_0$ ${}_2C_1$ ${}_2C_2$	1 2 1
$n = 3$ (3rd row)	${}_3C_0$ ${}_3C_1$ ${}_3C_2$ ${}_3C_3$	1 3 3 1
$n = 4$ (4th row)	${}_4C_0$ ${}_4C_1$ ${}_4C_2$ ${}_4C_3$ ${}_4C_4$	1 4 6 4 1
$n = 5$ (5th row)	${}_5C_0$ ${}_5C_1$ ${}_5C_2$ ${}_5C_3$ ${}_5C_4$ ${}_5C_5$	1 5 10 10 5 1

EXAMPLE 4 Use Pascal's triangle

SCHOOL CLUBS The 6 members of a Model UN club must choose 2 representatives to attend a state convention. Use Pascal's triangle to find the number of combinations of 2 members that can be chosen as representatives.

Solution

Because you need to find ${}_6C_2$, write the 6th row of Pascal's triangle by adding numbers from the previous row.

$n = 5$ (5th row)	1	5	10	10	5	1	
$n = 6$ (6th row)	1	6	15	20	15	6	1
	${}_6C_0$	${}_6C_1$	${}_6C_2$	${}_6C_3$	${}_6C_4$	${}_6C_5$	${}_6C_6$

▶ The value of ${}_6C_2$ is the third number in the 6th row of Pascal's triangle, as shown above. Therefore, ${}_6C_2 = 15$. There are 15 combinations of representatives for the convention.

✓ GUIDED PRACTICE for Example 4

6. **WHAT IF?** In Example 4, use Pascal's triangle to find the number of combinations of 2 members that can be chosen if the Model UN club has 7 members.

BINOMIAL EXPANSIONS There is an important relationship between powers of binomials and combinations. The numbers in Pascal's triangle can be used to find coefficients in binomial expansions. For example, the coefficients in the expansion of $(a + b)^4$ are the numbers of combinations in the row of Pascal's triangle for $n = 4$:

$$(a + b)^4 = \underset{{}_4C_0}{1}a^4 + \underset{{}_4C_1}{4}a^3b + \underset{{}_4C_2}{6}a^2b^2 + \underset{{}_4C_3}{4}ab^3 + \underset{{}_4C_4}{1}b^4$$

This result is generalized in the **binomial theorem**.

KEY CONCEPT *For Your Notebook*

Binomial Theorem

For any positive integer n, the binomial expansion of $(a + b)^n$ is:

$$(a + b)^n = {}_nC_0a^nb^0 + {}_nC_1a^{n-1}b^1 + {}_nC_2a^{n-2}b^2 + \cdots + {}_nC_na^0b^n$$

Notice that each term in the expansion of $(a + b)^n$ has the form ${}_nC_r\,a^{n-r}b^r$ where r is an integer from 0 to n.

EXAMPLE 5 Expand a power of a binomial sum

Use the binomial theorem to write the binomial expansion.

$$\begin{aligned}(x^2 + y)^3 &= {}_3C_0(x^2)^3y^0 + {}_3C_1(x^2)^2y^1 + {}_3C_2(x^2)^1y^2 + {}_3C_3(x^2)^0y^3\\ &= (1)(x^6)(1) + (3)(x^4)(y) + (3)(x^2)(y^2) + (1)(1)(y^3)\\ &= x^6 + 3x^4y + 3x^2y^2 + y^3\end{aligned}$$

POWERS OF BINOMIAL DIFFERENCES To expand a power of a binomial difference, you can rewrite the binomial as a sum. The resulting expansion will have terms whose signs alternate between + and −.

AVOID ERRORS

When a binomial has a term or terms with a coefficient other than 1, the coefficients of the binomial expansion are not the same as the corresponding row of Pascal's triangle.

EXAMPLE 6 Expand a power of a binomial difference

Use the binomial theorem to write the binomial expansion.

$$\begin{aligned}(a - 2b)^4 &= [a + (-2b)]^4\\ &= {}_4C_0a^4(-2b)^0 + {}_4C_1a^3(-2b)^1 + {}_4C_2a^2(-2b)^2 + {}_4C_3a^1(-2b)^3 + {}_4C_4a^0(-2b)^4\\ &= (1)(a^4)(1) + (4)(a^3)(-2b) + (6)(a^2)(4b^2) + (4)(a)(-8b^3) + (1)(1)(16b^4)\\ &= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4\end{aligned}$$

✓ **GUIDED PRACTICE** for Examples 5 and 6

Use the binomial theorem to write the binomial expansion.

7. $(x + 3)^5$ **8.** $(a + 2b)^4$ **9.** $(2p - q)^4$ **10.** $(5 - 2y)^3$

EXAMPLE 7 Find a coefficient in an expansion

Find the coefficient of x^4 in the expansion of $(3x + 2)^{10}$.

Solution

From the binomial theorem, you know the following:

$$(3x + 2)^{10} = {}_{10}C_0(3x)^{10}(2)^0 + {}_{10}C_1(3x)^9(2)^1 + \cdots + {}_{10}C_{10}(3x)^0(2)^{10}$$

Each term in the expansion has the form ${}_{10}C_r(3x)^{10-r}(2)^r$. The term containing x^4 occurs when $r = 6$:

$${}_{10}C_6(3x)^4(2)^6 = (210)(81x^4)(64) = 1{,}088{,}640x^4$$

▸ The coefficient of x^4 is 1,088,640.

✓ GUIDED PRACTICE for Example 7

11. Find the coefficient of x^5 in the expansion of $(x - 3)^7$.
12. Find the coefficient of x^3 in the expansion of $(2x + 5)^8$.

6.1 EXERCISES

HOMEWORK KEY

◯ = See **WORKED-OUT SOLUTIONS** Exs. 17, 29, and 49

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 35, 40, 41, and 52

SKILL PRACTICE

1. **VOCABULARY** Copy and complete: The binomial expansion of $(a + b)^n$ is given by the __?__.

2. ★ **WRITING** *Explain* the difference between permutations and combinations.

EXAMPLES 1, 2, and 3 for Exs. 3–18

COMBINATIONS **Find the number of combinations.**

3. ${}_5C_2$
4. ${}_{10}C_3$
5. ${}_9C_6$
6. ${}_8C_2$
7. ${}_{11}C_{11}$
8. ${}_{12}C_4$
9. ${}_7C_5$
10. ${}_{14}C_6$

ERROR ANALYSIS ***Describe*** **and correct the error in finding the number of combinations.**

11. $${}_6C_2 = \frac{6!}{(6-2)!} = \frac{720}{24} = 30$$ ✗

12. $${}_8C_3 = \frac{8!}{3!} = \frac{40{,}320}{6} = 6720$$ ✗

CARD HANDS **Find the number of possible 5-card hands that contain the cards specified. The cards are taken from a standard 52-card deck.**

13. 5 face cards (kings, queens, or jacks)
14. 4 kings and 1 other card
15. 1 ace and 4 cards that are not aces
16. 5 hearts or 5 diamonds
17. At most 1 queen
18. At least 1 spade

EXAMPLE 4 for Exs. 19–23

19. USING PATTERNS Copy Pascal's triangle and add rows for $n = 6, 7, 8, 9$, and 10.

PASCAL'S TRIANGLE **Use the rows of Pascal's triangle from Exercise 19 to write the binomial expansion.**

20. $(x + 3)^6$ **21.** $(y - 3z)^{10}$ **22.** $(a + b^2)^8$ **23.** $(2s - t^4)^7$

EXAMPLES 5 and 6 for Exs. 24–31

BINOMIAL THEOREM **Use the binomial theorem to write the binomial expansion.**

24. $(x + 2)^3$ **25.** $(c - 4)^5$ **26.** $(a + 3b)^4$ **27.** $(4p - q)^6$

28. $(w^3 - 3)^4$ **29.** $(2s^4 + 5)^5$ **30.** $(3u + v^2)^6$ **31.** $(x^3 - y^2)^4$

EXAMPLE 7 for Exs. 32–35

32. Find the coefficient of x^5 in the expansion of $(x - 2)^{10}$.

33. Find the coefficient of x^3 in the expansion of $(3x + 2)^5$.

34. Find the coefficient of x^6 in the expansion of $(x^2 - 3)^8$.

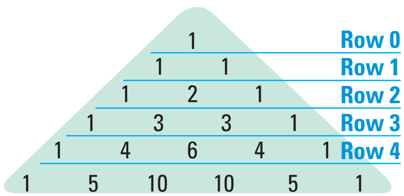
35. ★ **MULTIPLE CHOICE** Which is the coefficient of x^4 in the expansion of $(x - 3)^7$?

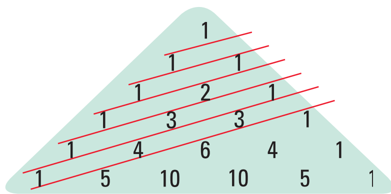
(A) −945 (B) −35 (C) −27 (D) 2835

PASCAL'S TRIANGLE **In Exercises 36 and 37, use the diagrams shown.**

36. What is the sum of the numbers in each of rows 0–4 of Pascal's triangle? What is the sum in row n?

37. *Describe* the pattern formed by the sums of the numbers along the diagonal segments of Pascal's triangle.





REASONING **In Exercises 38 and 39, decide whether the problem requires *combinations* or *permutations* to find the answer. Then solve the problem.**

38. NEWSPAPER Your school newspaper has an editor-in-chief and an assistant editor-in-chief. The staff of the newspaper has 12 students. In how many ways can students be chosen for these two positions?

39. STUDENT COUNCIL Five representatives from a senior class of 280 students are to be chosen for the student council. In how many ways can students be chosen to represent the senior class on the student council?

40. ★ **MULTIPLE CHOICE** A relay race has a team of 4 runners who run different parts of the race. There are 20 students on your track squad. In how many ways can the coach select students to compete on the relay team?

(A) 4845 (B) 40,000 (C) 116,280 (D) 160,000

41. ★ **SHORT RESPONSE** *Explain* how the formula for ${}_nC_n$ suggests the definition $0! = 1$.

CHALLENGE **Verify the identity. *Justify* each of your steps.**

42. ${}_nC_0 = 1$ **43.** ${}_nC_n = 1$ **44.** ${}_nC_r \cdot {}_rC_m = {}_nC_m \cdot {}_{n-m}C_{r-m}$

45. ${}_nC_1 = {}_nP_1$ **46.** ${}_nC_r = {}_nC_{n-r}$ **47.** ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$

PROBLEM SOLVING

EXAMPLES 1, 2, and 3 for Exs. 48–50

48. MUSIC You want to purchase 3 CDs from an online collection that contains the types of music shown at the right. You want each CD to contain a different type of music such that 2 CDs are different types of contemporary music and 1 CD is a type of classical music. How many different sets of music types can you choose?

49. FLOWERS You are buying a bouquet. The florist has 18 types of flowers that you can use to make the bouquet. You want to use *exactly* 3 types of flowers. How many different combinations of flower types can you use in your bouquet?

50. ARCADE GAMES An arcade has 20 different arcade games. You want to play at least 14 of them. How many different combinations of arcade games can you play?

51. MULTI-STEP PROBLEM A televised singing competition picks a winner from 20 original contestants over the course of five episodes. During each of the first, second, and third episodes, 5 singers are eliminated by the end of the episode. The fourth episode eliminates 2 more singers, and the winner is selected at the end of the fifth episode.

a. How many combinations of 5 singers out of the original 20 can be eliminated during the first episode?

b. How many combinations of 5 singers out of the 15 singers who started the second episode can be eliminated during the second episode?

c. How many combinations of singers can be eliminated during the third episode? during the fourth episode? during the fifth episode?

d. Find the total number of ways in which the 20 original contestants can be eliminated to produce a winner.

52. ★ EXTENDED RESPONSE A group of 15 high school students is volunteering at a local fire station. Of these students, 5 will be assigned to wash fire trucks, 7 will be assigned to repaint the station's interior, and 3 will be assigned to do maintenance on the station's exterior.

a. Calculate One way to count the number of possible job assignments is to find the number of permutations of 5 W's (for "wash"), 7 R's (for "repainting"), and 3 M's (for "maintenance"). Use this method to write the number of possible job assignments first as an expression involving factorials and then as a number.

b. Calculate Another way to count the number of possible job assignments is to first choose the 5 W's, then choose the 7 R's, and then choose the 3 M's. Use this method to write the number of possible job assignments first as an expression involving factorials and then as a number.

c. Analyze *Compare* your results from parts (a) and (b). *Explain* why they make sense.

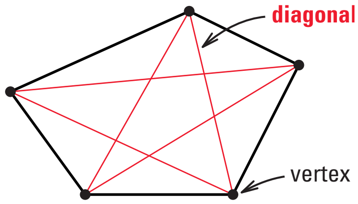
Volunteers in Aniak, Alaska

AP Images/Marc Lester

○ = See **WORKED-OUT SOLUTIONS** in Student Resources

★ = **STANDARDIZED TEST PRACTICE**

53. **CHALLENGE** A polygon is *convex* if no line that contains a side of the polygon contains a point in the interior of the polygon. Consider a convex polygon with n sides.



a. Use the combinations formula to write an expression for the number of line segments that join pairs of vertices on an n-sided polygon.

b. Use your result from part (a) to write a formula for the number of diagonals of an n-sided convex polygon.

Use a Simulation to Test an Assumption

MATHEMATICAL PRACTICES

Use appropriate tools strategically.

MATERIALS • coins • graphing calculator

QUESTION How do you determine whether a coin is fair?

How do you know if a coin is "fair"? That is, when you flip it, how do you know a coin is equally likely to land heads or tails?

You can flip an actual coin many times to help you decide whether you think it's fair. But what kinds of seemingly "unusual" outcomes might occur even with a fair coin? How do you know what you can expect?

In this activity, you will perform physical experiments with a coin and also use *simulation* with a graphing calculator to model flipping a coin. The simulation lets you quickly repeat an event with two equally likely outcomes to compare with results that you might get from an actual coin.

EXPLORE 1 Perform an experiment

A friend gives you a coin. You flip it 4 times, and you get tails all 4 times. Would you feel confident in concluding that the coin is not fair?

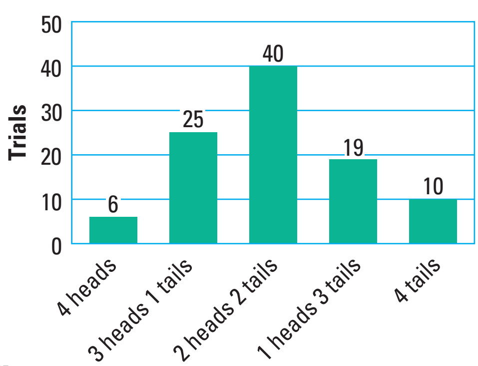
STEP 1 ***Flip coins***

Working in pairs, flip a coin 4 times. Repeat this experiment a total of 10 times, so that you have 10 trials of 4 flips. Record your results in a table like the one shown. Count the number of trials that fall into each of these categories: 4 heads, 3 heads and 1 tails, 2 heads and 2 tails, 1 heads and 3 tails, 4 tails.

	Flip 1	Flip 2	Flip 3	Flip 4
Trial 1	?	?	?	?
Trial 2	?	?	?	?

STEP 2 ***Display results***

Collect all of the data generated by the pairs of students in your class. Combine the outcomes for all of the trials, and then display the class data using a bar graph set up like the one at the right.



STEP 3 ***Analyze results***

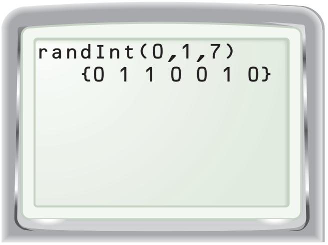
What outcome or outcomes occurred the most frequently? How many times did 4 heads or 4 tails occur? Would you be reasonably certain concluding that the coin your friend gave you is not fair? *Explain.*

EXPLORE 2 Perform a simulation of an experiment

A friend gives you a coin. You flip it 7 times, and you get tails all 7 times. Would you feel confident in concluding that the coin is not fair?

STEP 1 *Generate data*

For the simulation, let 0 represent flipping heads and let 1 represent flipping tails. On a graphing calculator, press the MATH key, select the "PRB" menu, and choose "randInt(." Perform the keystrokes for "randInt(0, 1, 7)" and press ENTER. The calculator will produce a list of seven digits (0 or 1) at random, as shown, where the outcomes 0 and 1 are equally likely to occur.



Press ENTER 20 times to obtain 20 lists of 7 "coin flips." Keep a tally to record the number of these 20 trials whose results fall into each of these categories: 7 heads, 6 heads and 1 tail, 5 heads and 2 tails, ..., 1 heads and 6 tails, 7 tails.

STEP 2 *Display results*

Collect all of the data generated by the pairs in your class. Combine the outcomes for all of the trials and display the class data using a bar graph similar to the one used in Explore 1. It will have 8 bars.

STEP 3 *Analyze results*

What outcome or outcomes occurred the most? How many times did 7 heads or 7 tails occur? Would you be reasonably certain concluding that the coin your friend gave you is not fair? *Explain.*

DRAW CONCLUSIONS Use your observations to complete these exercises

1. In Explore 1, find the theoretical probability of flipping 4 tails in a row. Does this affect your confidence in the coin's fairness or unfairness? *Explain.*
2. In Explore 2, find the theoretical probability of flipping 7 tails in a row. Does this affect your confidence in the coin's fairness or unfairness? *Explain.*
3. A simulation of flipping a coin 5 times in a row is performed 200 times. The results of the simulation are shown in the table below.

	5 heads 0 tails	4 heads 1 tails	3 heads 2 tails	2 heads 3 tails	1 heads 4 tails	0 heads 5 tails
Outcomes	1	5	27	61	72	34

   a. Make a bar graph of the data.

   b. Do you think that the simulation represents a coin that is fair or not fair? *Explain.*