6.2 Construct and Interpret Binomial Distributions

AP Images/John Russell

Before You found probabilities of events.

Now You will study probability distributions.

Why? So you can describe interest in museums, as in Ex. 46.

Key Vocabulary
- **random variable**
- **probability distribution**
- **binomial distribution**
- **binomial experiment**
- **symmetric**
- **skewed**

COMMON CORE

CC.9-12.S.MD.3 (+) Develop a probability distribution for a random variable defined for a sample space in which theoretical probabilities can be calculated; find the expected value.*

A **random variable** is a variable whose value is determined by the outcomes of a random event. For example, when you roll a six-sided die, you can define a random variable X that represents the number showing on the die. So, the possible values of X are 1, 2, 3, 4, 5, and 6. For every random variable, a *probability distribution* can be defined.

KEY CONCEPT *For Your Notebook*

Probability Distributions

A **probability distribution** is a function that gives the probability of each possible value of a random variable. The sum of all the probabilities in a probability distribution must equal 1.

Probability Distribution for Rolling a Die

X	1	2	3	4	5	6
$P(X)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

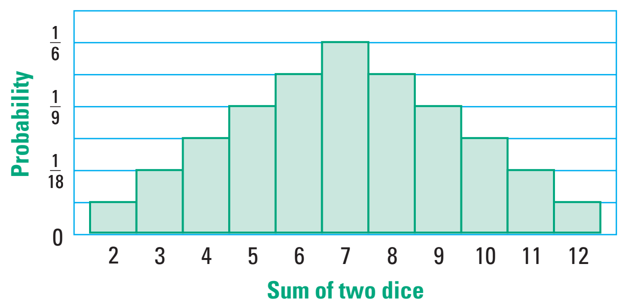
EXAMPLE 1 Construct a probability distribution

Let X be a random variable that represents the sum when two six-sided dice are rolled. Make a table and a histogram showing the probability distribution for X.

REVIEW COMPOUND EVENTS
Recall that there are 36 possible outcomes when rolling two six-sided dice. These are listed in Example 4.

Solution

The possible values of X are the integers from 2 to 12. The table shows how many outcomes of rolling two dice produce each value of X. Divide the number of outcomes for X by 36 to find $P(X)$.



X (sum)	2	3	4	5	6	7	8	9	10	11	12
Outcomes	1	2	3	4	5	6	5	4	3	2	1
$P(X)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

EXAMPLE 2 Interpret a probability distribution

Use the probability distribution in Example 1 to answer each question.

a. What is the most likely sum when rolling two six-sided dice?

b. What is the probability that the sum of the two dice is at least 10?

Solution

a. The most likely sum when rolling two six-sided dice is the value of X for which $P(X)$ is greatest. This probability is greatest for $X = 7$. So, the most likely sum when rolling the two dice is 7.

b. The probability that the sum of the two dice is at least 10 is:

$$P(X \geq 10) = P(X = 10) + P(X = 11) + P(X = 12)$$

$$= \frac{3}{36} + \frac{2}{36} + \frac{1}{36}$$

$$= \frac{6}{36}$$

$$= \frac{1}{6}$$

$$\approx 0.167$$

✓ GUIDED PRACTICE for Examples 1 and 2

A tetrahedral die has four sides numbered 1 through 4. Let X be a random variable that represents the sum when two such dice are rolled.

1. Make a table and a histogram showing the probability distribution for X.

2. What is the most likely sum when rolling the two dice? What is the probability that the sum of the two dice is at most 3?

BINOMIAL DISTRIBUTIONS One type of probability distribution is a **binomial distribution**. A binomial distribution shows the probabilities of the outcomes of a *binomial experiment*.

KEY CONCEPT *For Your Notebook*

Binomial Experiments

A **binomial experiment** meets the following conditions:

- There are n independent trials.
- Each trial has only two possible outcomes: success and failure.
- The probability of success is the same for each trial. This probability is denoted by p. The probability of failure is given by $1 - p$.

For a binomial experiment, the probability of exactly k successes in n trials is:

$$P(k \text{ successes}) = {}_nC_k p^k (1 - p)^{n-k}$$

EXAMPLE 3 Construct a binomial distribution

SPORTS SURVEYS According to a survey, about 41% of U.S. households have a soccer ball. Suppose you ask 6 randomly chosen U.S. households whether they have a soccer ball. Draw a histogram of the binomial distribution for your survey.

Solution

The probability that a randomly selected household has a soccer ball is $p = 0.41$. Because you survey 6 households, $n = 6$.

AVOID ERRORS
You can check your calculations for a binomial distribution by adding all the probabilities. The sum should always be 1.

$P(k = 0) = {}_6C_0(0.41)^0(0.59)^6 \approx 0.042$

$P(k = 1) = {}_6C_1(0.41)^1(0.59)^5 \approx 0.176$

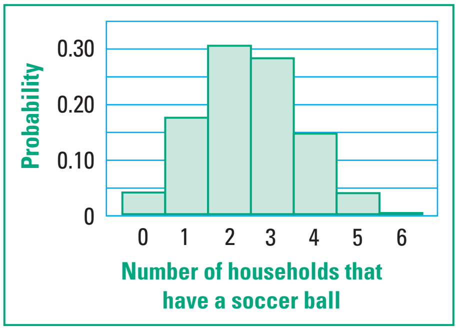
$P(k = 2) = {}_6C_2(0.41)^2(0.59)^4 \approx 0.306$

$P(k = 3) = {}_6C_3(0.41)^3(0.59)^3 \approx 0.283$

$P(k = 4) = {}_6C_4(0.41)^4(0.59)^2 \approx 0.148$

$P(k = 5) = {}_6C_5(0.41)^5(0.59)^1 \approx 0.041$

$P(k = 6) = {}_6C_6(0.41)^6(0.59)^0 \approx 0.005$



A histogram of the distribution is shown.

Animated Algebra at my.hrw.com

EXAMPLE 4 Interpret a binomial distribution

Use the binomial distribution in Example 3 to answer each question.

a. What is the most likely outcome of the survey?

b. What is the probability that at most 2 households have a soccer ball?

Solution

a. The most likely outcome of the survey is the value of k for which $P(k)$ is greatest. This probability is greatest for $k = 2$. So, the most likely outcome is that 2 of the 6 households have a soccer ball.

b. The probability that at most 2 households have a soccer ball is:

$$P(k \le 2) = P(k = 2) + P(k = 1) + P(k = 0)$$
$$\approx 0.306 + 0.176 + 0.042$$
$$\approx 0.524$$

▶ So, the probability is about 52%.

✓ GUIDED PRACTICE for Examples 3 and 4

In Sweden, 61% of households have a soccer ball. Suppose you ask 6 randomly chosen Swedish households whether they have a soccer ball.

3. Draw a histogram showing the binomial distribution for your survey.

4. What is the most likely outcome of your survey? What is the probability that at most 2 households you survey have a soccer ball?

CLASSIFY DISTRIBUTIONS Note that the distribution in Example 1 is symmetric, while the distribution in Example 3 is skewed.

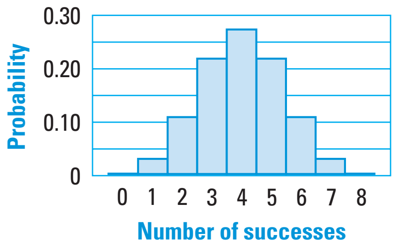
SYMMETRIC AND SKEWED DISTRIBUTIONS Suppose a probability distribution is represented by a histogram. The distribution is **symmetric** if you can draw a vertical line that divides the histogram into two parts that are mirror images. A distribution that is *not* symmetric is called **skewed**.

EXAMPLE 5 Classify distributions as symmetric or skewed

Describe the shape of the binomial distribution that shows the probability of exactly *k* successes in 8 trials if (a) *p* = 0.5 and (b) *p* = 0.9.

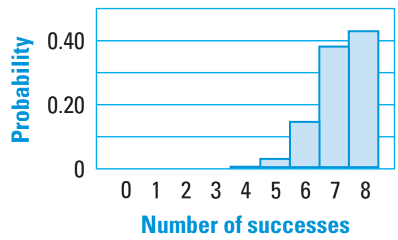
Solution

a.



Symmetric; the left half is a mirror image of the right half.

b.



Skewed; the distribution is not symmetric about any vertical line.

✓ GUIDED PRACTICE for Example 5

5. A binomial experiment consists of 5 trials with probability *p* of success on each trial. Describe the shape of the binomial distribution that shows the probability of exactly *k* successes if (**a**) $p = 0.4$ and (**b**) $p = 0.5$.

6.2 EXERCISES

HOMEWORK KEY

○ = See **WORKED-OUT SOLUTIONS** Exs. 5, 21, and 45

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 9, 32, 39, and 48

◆ = **MULTIPLE REPRESENTATIONS** Ex. 47

SKILL PRACTICE

1. VOCABULARY Copy and complete: A probability distribution represented by a histogram is __?__ if you can draw a vertical line dividing the histogram into two parts that are mirror images.

2. ★ WRITING *Explain* the difference between a binomial experiment and a binomial distribution.

EXAMPLE 1 for Exs. 3–5

CONSTRUCTING PROBABILITY DISTRIBUTIONS **Make a table and a histogram showing the probability distribution for the random variable.**

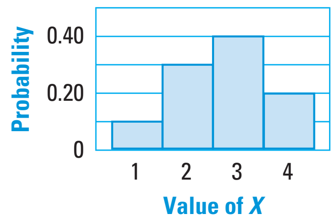
3. X = the number on a table tennis ball randomly chosen from a bag that contains 5 balls labeled "1," 3 balls labeled "2," and 2 balls labeled "3."

4. W = 1 if a randomly chosen letter is A, E, I, O, or U and 2 otherwise.

5. N = the number of digits in a random integer from 0 through 999.

EXAMPLE 2
for Exs. 6–9

INTERPRETING PROBABILITY DISTRIBUTIONS **In Exercises 6–9, use the given histogram of a probability distribution for a random variable *X*.**



6. What is the probability that *X* is equal to 1?

7. What is the most likely value for *X*?

8. What is the probability that *X* is odd?

9. ★ **MULTIPLE CHOICE** What is the probability that *X* is at least 3?

Ⓐ 0.2 Ⓑ 0.4 Ⓒ 0.6 Ⓓ 0.8

EXAMPLES 3 and 4
for Exs. 10–32

CALCULATING PROBABILITIES **Calculate the probability of tossing a coin 20 times and getting the given number of heads.**

10. 1 **11.** 2 **12.** 4 **13.** 6

14. 9 **15.** 12 **16.** 15 **17.** 18

BINOMIAL PROBABILITIES **Calculate the probability of randomly guessing the given number of correct answers on a 30-question multiple choice exam that has choices A, B, C, and D for each question.**

18. 0 **19.** 2 **20.** 6 **21.** 11

22. 15 **23.** 21 **24.** 26 **25.** 30

ERROR ANALYSIS ***Describe*** **and correct the error in calculating the probability of rolling a 1 exactly 3 times in 5 rolls of a six-sided die.**

26.

$$P(k = 3) = {}_5C_3\left(\frac{1}{6}\right)^{5-3}\left(\frac{5}{6}\right)^3$$
$$\approx 0.161$$

27.

$$P(k = 3) = \left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{5-3}$$
$$\approx 0.003$$

BINOMIAL DISTRIBUTIONS **Calculate the probability of *k* successes for a binomial experiment consisting of *n* trials with probability *p* of success on each trial.**

28. $k \le 3, n = 7, p = 0.3$ **29.** $k \ge 5, n = 8, p = 0.6$

30. $k \le 2, n = 5, p = 0.12$ **31.** $k \ge 10, n = 15, p = 0.75$

32. ★ **MULTIPLE CHOICE** You perform a binomial experiment consisting of 10 trials with a probability of success of 36% on each trial. What is the most likely number of successes?

Ⓐ 3 Ⓑ 4 Ⓒ 6 Ⓓ 7

EXAMPLE 5
for Exs. 33–38

HISTOGRAMS **A binomial experiment consists of *n* trials with probability *p* of success on each trial. Draw a histogram of the binomial distribution that shows the probability of exactly *k* successes. *Describe* the distribution as either *symmetric* or *skewed*. Then find the most likely number of successes.**

33. $n = 3, p = 0.3$ **34.** $n = 6, p = 0.5$ **35.** $n = 4, p = 0.16$

36. $n = 7, p = 0.85$ **37.** $n = 8, p = 0.025$ **38.** $n = 12, p = 0.5$

39. ★ **OPEN-ENDED MATH** Construct a symmetric probability distribution for a random variable *X* and a skewed probability distribution for a random variable *Y*. Make a table and a histogram for each distribution.

In Exercises 40–42, you will derive the binomial probability formula. Consider a binomial experiment with n trials and probability p of success on each trial.

40. For any particular sequence of k successes and $n - k$ failures, what is the probability that the sequence occurs? *Explain.*

41. How many sequences of k successes and $n - k$ failures are there? *Explain.*

42. **CHALLENGE** Use your results from Exercises 40 and 41 to justify the binomial probability formula.

PROBLEM SOLVING

EXAMPLES 3 and 4 for Exs. 43–46

43. **HEALTH** About 1% of people are allergic to bee stings. What is the probability that exactly 1 person in a class of 25 is allergic to bee stings?

44. **BASKETBALL** Predrag Stojakovic of the Sacramento Kings made 92.7% of his free throw attempts in the 2003–2004 NBA regular season. What is the probability that he will make exactly 10 of his next 15 free throw attempts?

45. **BLOOD TYPE** The chart shows the distribution of blood types (O, A, B, AB) and Rh factor ($^+$ or $^-$) for human blood. If, at random, 10 people donate blood to a blood bank during a certain hour, find the probability of each event.

Percent of Population by Blood Type							
O^+	O^-	A^+	A^-	B^+	B^-	AB^+	AB^-
37%	6%	34%	6%	10%	2%	4%	1%

a. Exactly 5 of the people are type A^+.

b. Exactly 2 of the people are Rh^-.

c. At most 2 of the people are type O.

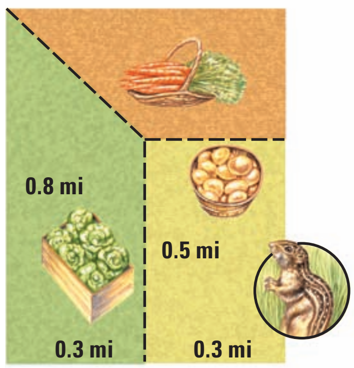
d. At least 5 of the people are Rh^+.

46. **FINE ARTS** A survey states that 35% of people in the United States visited an art museum in a certain year. You randomly select 10 U.S. citizens.

a. Draw a histogram showing the binomial distribution of the number of people who visited an art museum.

b. What is the probability that at most 4 people visited an art museum?

47. **MULTIPLE REPRESENTATIONS** An average of 7 gopher holes appear on the farm shown each week. Let X represent how many of the 7 gopher holes appear in the carrot patch. Assume that a gopher hole has an equal chance of appearing at any point on the farm.



a. **Calculating Probabilities** Find $P(X)$ for $X = 0, 1, 2, \ldots, 7$.

b. **Making a Table** Make a table showing the probability distribution for X.

c. **Making a Histogram** Make a histogram showing the probability distribution for X.

48. ★ **EXTENDED RESPONSE** Assume that having a male child and having a female child are independent events and that the probability of each is 0.5.

a. A couple has 4 male children. Evaluate the validity of this statement: "The first 4 kids were all boys, so the next one will probably be a girl."

b. What is the probability of having 4 male children and then a female child?

c. Let X be a random variable that represents the number of children a couple already has when they have their first female child. Draw a histogram of the distribution of $P(X)$ for $0 \le X \le 10$ and describe its shape.

49. **CHALLENGE** An entertainment system has n speakers. Each speaker will function properly with probability p, independent of whether the other speakers are functioning. The system will operate effectively if at least 50% of its speakers are functioning. For what values of p is a 5-speaker system more likely to operate than a 3-speaker system?

QUIZ

Find the number of combinations.

1. $_8C_6$ 2. $_7C_4$ 3. $_9C_0$ 4. $_{12}C_{11}$

Use the binomial theorem to write the binomial expansion.

5. $(x + 5)^5$ 6. $(2s - 3)^6$ 7. $(3u + v)^4$ 8. $(2x^3 - 3y)^5$

9. Find the coefficient of x^3 in the expansion of $(x + 2)^9$.

10. **MENU CHOICES** A pizza parlor runs a special where you can buy a large pizza with 1 cheese, 1 vegetable, and 2 meats for \$12. You have a choice of 5 cheeses, 10 vegetables, and 6 meats. How many different variations of the pizza special are possible?

Calculate the probability of getting the given number of 6's when rolling a six-sided die 10 times.

11. 0 12. 1 13. 4 14. 8

A binomial experiment consists of n trials with probability p of success on each trial. Draw a histogram of the binomial distribution that shows the probability of exactly k successes.

15. $n = 5, p = 0.2$ 16. $n = 8, p = 0.5$ 17. $n = 6, p = 0.72$

PROBLEM SOLVING WORKSHOP
LESSON 6.2

Using ALTERNATIVE METHODS

Another Way to Solve Examples 3

MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them.

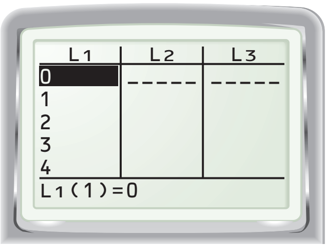
MULTIPLE REPRESENTATIONS In Example 3, you constructed a binomial distribution. Some calculators have a binomial probability distribution function that you can use to calculate binomial probabilities. You can then use the calculator to draw a histogram of the distribution.

PROBLEM 1

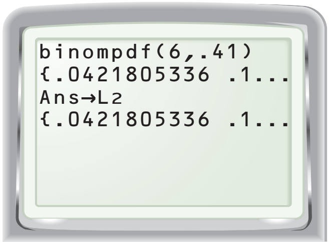
SPORTS SURVEYS According to a survey, about 41% of U.S. households have a soccer ball. Suppose you ask 6 randomly chosen U.S. households whether they have a soccer ball. Draw a histogram of the binomial distribution for your survey.

METHOD

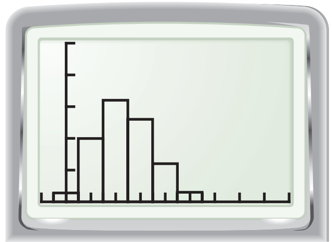
STEP 1 **Enter** values of k. Let $p = 0.41$ be the probability that a randomly selected household has a soccer ball. Enter the k-values 0 through 6 into list L_1 on the graphing calculator.



STEP 2 **Find** values of $P(k)$. Enter the binomial probability command to generate $P(k)$ for all seven k-values. Store the results in list L_2.



STEP 3 **Draw** histogram. Set up the histogram to use the numbers in list L_1 as x-values and the numbers in list L_2 as frequencies. Draw the histogram in a suitable viewing window.



PRACTICE

A binomial experiment consists of n trials with probability p of success on each trial. Use a graphing calculator to draw a histogram of the binomial distribution that shows the probability of exactly k successes. Then find the most likely number of successes.

1. $n = 12, p = 0.29$
2. $n = 14, p = 0.58$
3. $n = 15, p = 0.805$
4. **WHAT IF?** In the example, how do your histogram and the most likely number of adults change if you survey 14 adults at random?

MIXED REVIEW *of Problem Solving*

MATHEMATICAL PRACTICES Make sense of problems and persevere in solving them.

1. **MULTI-STEP PROBLEM** About 71% of households in Alonso's town have a pet. Alonso is tasked with finding the probability distribution for a survey of 7 random households. Part of his calculation is given:

$$P(k = 3) = {}_7C_3(0.71)^3(0.29)^4 < 0.089$$

   a. Explain what $P(k = 3)$ represents in this calculation.

   b. Explain what ${}_7C_3$ represents in this calculation.

   c. Explain what $(0.71)^3$ represents in this calculation.

   d. Explain what $(0.29)^4$ represents in this calculation.

2. **MULTI-STEP PROBLEM** According to a survey, 62% of U.S. adults consider themselves sports fans. You randomly select 14 adults to survey.

   a. Draw a histogram of the binomial distribution showing the probability that k adults consider themselves sports fans.

   b. What is the most likely number of adults who consider themselves sports fans?

   c. What is the probability that at least 7 adults consider themselves sports fans?

3. **SHORT RESPONSE** A community theater is presenting a series of 15 operas this summer. Melinda wants to attend at least 4 of them. Write a subtractive expression using notation representing the number of different combinations of operas she can attend, and then find that number.

4. **GRIDDED ANSWER** You want to make a fruit smoothie using 3 of the fruits listed. How many different fruit smoothies can you make?

5. **GRIDDED ANSWER** A softball player gets a hit in about 31% of her at-bats. You randomly select 15 of the player's at-bats. What is the most likely number of hits the player will have in those at bats?

6. **SHORT RESPONSE** You must take 18 elective courses to meet your graduation requirements for college. There are 30 courses that you are interested in. Does finding the number of possible course selections involve *permutations* or *combinations*? *Explain*. How many different course selections are possible?

7. **OPEN-ENDED** Give an example of a real-life problem for which the answer is the product of two combinations. Provide a solution.

8. **GRIDDED ANSWER** An ice cream shop offers a choice of 31 flavors. How many different ice cream cones can be made with three scoops of ice cream if each scoop is a different flavor and the order of the scoops is not important?

Investigate the Shapes of Data Distributions

MATHEMATICAL PRACTICES

Use appropriate tools strategically.

MATERIALS • graph paper • graphing calculator

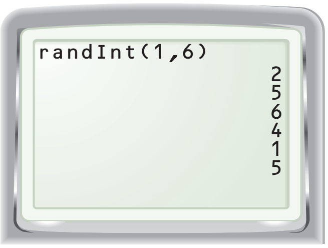
QUESTION How do you describe and interpret data distributions with various shapes?

A histogram of data can have different shapes. It may be symmetrical or not, may be bell-shaped or not, or may have a long "tail" in one direction. It may have no recognizable shape. There are recurring patterns in data distributions, however, that let you predict probabilities involving data values.

EXPLORE 1 Simulate rolling a die repeatedly until you get a 5

STEP 1 Perform a simulation

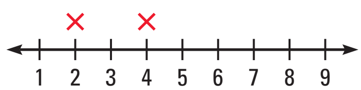
Press MATH, then select "PRB," then "randInt(." The command randInt(1, 6) gives a random integer from 1 to 6 when you press ENTER. The screen shows 6 results.



STEP 2 Make a plot

In Step 1, a "5" first appears in the 2nd trial. It takes 4 more trials to get the next "5." These two results are shown below. Make a similar plot for your data. Plot 40 or 50 results.

Number of trials to get a 5



STEP 3 Observe and compare

Describe your plot. Include the following.

- What is the general shape?
- Is it symmetric?
- Does it have a clear "middle?"

Compare your plot with those of other groups. Were their results similar?

What was the greatest number of rolls required by any single group to get a 5?

EXPLORE 2 Simulate rolling a die 100 times

STEP 1 Perform a simulation

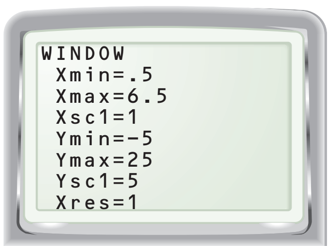
To simulate 100 rolls of a die, you can perform all 100 "rolls" at once and store the results in a list by entering the command below.

randInt(1, 6, 100) STO ▶
2nd [L1]

Pressing ENTER displays the list on the home screen and enters it in the Statistics memory.

STEP 2 Make a histogram

Press 2nd [Stat Plot], turn on Plot1, and select the histogram icon. Set "Xlist" to L1 and "Freq" to 1. Enter the window below, then press GRAPH.



STEP 3 Observe and compare

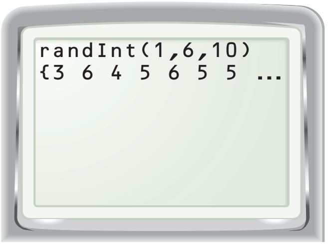
Describe your plot and compare plots as you did in Step 3 above.

You can very quickly make a new plot:

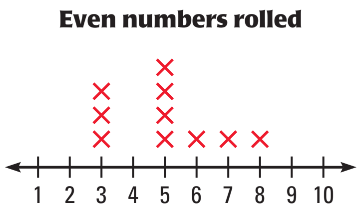
Press 2nd [QUIT] to return to the home screen. Press ENTER to produce a new list. Then press GRAPH. Do this a few times. How does the graph change?

EXPLORE 3 Simulate rolling 10 dice and counting how many display "even results"

STEP 1 ***Perform a simulation*** Enter randInt(1, 6, 10) and press ENTER to simulate rolling 10 dice. Below you can see three "dice" that display even results. (To see the last three results in the list, scroll to the right.)



STEP 2 ***Make a plot*** Press ENTER, count the number of even numbers in the new list, and record it in a plot like the one below, which shows results for 10 rolls of 10 dice. Repeat as many times as time allows.



STEP 3 ***Observe and compare*** Describe your plot and compare plots as you did in Explore 1 and 2.

How does the shape of this plot compare with the shapes of the plots from Explore 1 and 2?

EXPLORE 4 Combine data

STEP 1 ***Combine results from Explore 1*** Combine the results of all groups to make a class histogram. Compare your plot and the class histogram.

STEP 2 ***Simulate combining results from Explore 2*** To simulate combining results from Explore 2 for 10 groups, repeat the first two steps of Explore 2, but use the command randInt(1, 6, 999). (Note that 999 is the largest number the calculator allows.) Also, change the viewing window to Ymin = −50, Ymax = 250, and Yscl = 50. Compare your histogram and the class histogram.

STEP 3 ***Combine results from Explore 3*** Combine the results of all groups to make a class histogram. Compare your plot and the class histogram.

DRAW CONCLUSIONS Use your observations to complete these exercises

1. Did the graphs in each Explore have similar or very different shapes? *Explain.*
2. For each Explore, how did combining the results of the groups into one graph change the shapes? Predict what you think the shape of each histogram might be if you could repeat each simulation thousands of times.
3. Symmetric, bell-shaped data distributions, such as the one at the right, occur frequently in real world models. This curve has special properties that make it extremely useful for finding probabilities. Which of the Explore results most closely resembles a curve with this shape?