6.3 Use Normal Distributions

Before You interpreted probability distributions.

Now You will study normal distributions.

Why? So you can model animal populations, as in Example 3.

Key Vocabulary
- **normal distribution**
- **normal curve**
- **standard normal distribution**
- ***z*-score**

COMMON CORE

CC.9-12.S.ID.4 Use the mean and standard deviation of a data set to fit it to a normal distribution and to estimate population percentages. Recognize that there are data sets for which such a procedure is not appropriate. Use calculators, spreadsheets, and tables to estimate areas under the normal curve.*

INTERPRET GRAPHS

An area under a normal curve can be interpreted either as a percentage of the data values in the distribution or as a probability.

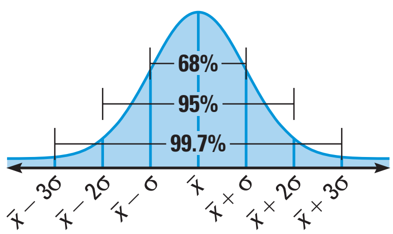
You have studied probability distributions. One type of probability distribution is a *normal distribution*. A **normal distribution** is modeled by a bell-shaped curve called a **normal curve** that is symmetric about the mean.

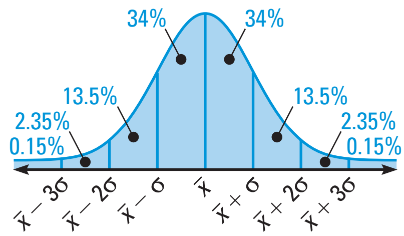
KEY CONCEPT — *For Your Notebook*

Areas Under a Normal Curve

A normal distribution with mean $\overline{x}$ and standard deviation σ has the following properties:

- The total area under the related normal curve is 1.
- About 68% of the area lies within 1 standard deviation of the mean.
- About 95% of the area lies within 2 standard deviations of the mean.
- About 99.7% of the area lies within 3 standard deviations of the mean.



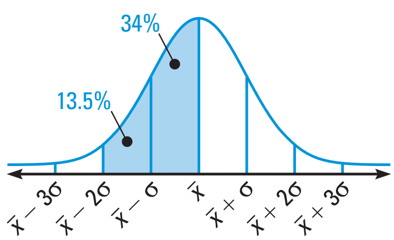


EXAMPLE 1 Find a normal probability

A normal distribution has mean $\overline{x}$ and standard deviation σ. For a randomly selected x-value from the distribution, find $P(\overline{x} - 2\sigma \le x \le \overline{x})$.

Solution

The probability that a randomly selected x-value lies between between $\overline{x} - 2\sigma$ and $\overline{x}$ is the shaded area under the normal curve shown.



$$P(\overline{x} - 2\sigma \le x \le \overline{x}) = 0.135 + 0.34 = 0.475$$

Animated Algebra at my.hrw.com

EXAMPLE 2 Interpret normally distributed data

READING
The abbreviation "mg/dl" stands for "milligrams per deciliter."

HEALTH The blood cholesterol readings for a group of women are normally distributed with a mean of 172 mg/dl and a standard deviation of 14 mg/dl.

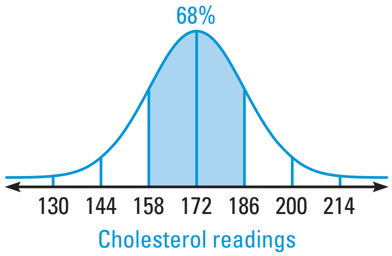
a. About what percent of the women have readings between 158 and 186?

b. Readings less than 158 are considered desirable. About what percent of the readings are desirable?

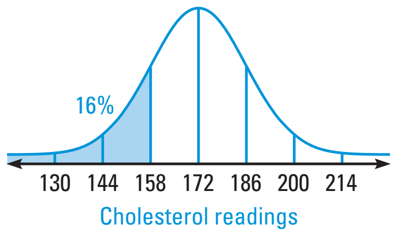
USE PERCENTILES
When *n*% of the data are less than or equal to a certain value, that value is called the *n*th *percentile.* Part (b) of Example 2 shows that 158 is the 16th percentile. Similarly, 172 is the 50th percentile.

Solution

a. The readings of 158 and 186 represent one standard deviation on either side of the mean, as shown below. So, 68% of the women have readings between 158 and 186.



b. A reading of 158 is one standard deviation to the left of the mean, as shown. So, the percent of readings that are desirable is 0.15% + 2.35% + 13.5%, or 16%.



✓ GUIDED PRACTICE for Examples 1 and 2

A normal distribution has mean $\overline{x}$ and standard deviation σ. Find the indicated probability for a randomly selected *x*-value from the distribution.

1. $P(x \le \overline{x})$

2. $P(x \ge \overline{x})$

3. $P(\overline{x} \le x \le \overline{x} + 2\sigma)$

4. $P(\overline{x} - \sigma \le x \le \overline{x})$

5. $P(x \le \overline{x} - 3\sigma)$

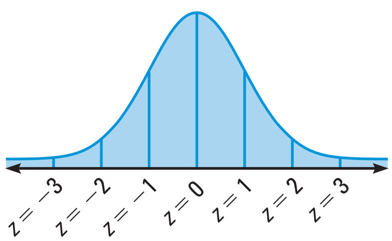
6. $P(x \ge \overline{x} + \sigma)$

7. WHAT IF? In Example 2, what percent of the women have readings between 172 and 200?

STANDARD NORMAL DISTRIBUTION The **standard normal distribution** is the normal distribution with mean 0 and standard deviation 1. The formula below can be used to transform *x*-values from a normal distribution with mean $\overline{x}$ and standard deviation σ into *z*-values having a standard normal distribution.

Formula: $z = \frac{x - \overline{x}}{\sigma}$

Subtract the mean from the given *x*-value, then divide by the standard deviation.



The *z*-value for a particular *x*-value is called the ***z*-score** for the *x*-value and is the number of standard deviations the *x*-value lies above or below the mean $\overline{x}$.

STANDARD NORMAL TABLE If z is a randomly selected value from a standard normal distribution, you can use the table below to find the probability that z is less than or equal to some given value. For example, the table shows that $P(z \le -0.4) = 0.3446$. You can find the value of $P(z \le -0.4)$ in the table by finding the value where row **−0** and column **.4** intersect.

READING
In the table, the value .0000+ means "slightly more than 0" and the value 1.0000− means "slightly less than 1."

Standard Normal Table

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
−3	.0013	.0010	.0007	.0005	.0003	.0002	.0002	.0001	.0001	.0000+
−2	.0228	.0179	.0139	.0107	.0082	.0062	.0047	.0035	.0026	.0019
−1	.1587	.1357	.1151	.0968	.0808	.0668	.0548	.0446	.0359	.0287
−0	.5000	.4602	.4207	.3821	.3446	.3085	.2743	.2420	.2119	.1841
0	.5000	.5398	.5793	.6179	.6554	.6915	.7257	.7580	.7881	.8159
1	.8413	.8643	.8849	.9032	.9192	.9332	.9452	.9554	.9641	.9713
2	.9772	.9821	.9861	.9893	.9918	.9938	.9953	.9965	.9974	.9981
3	.9987	.9990	.9993	.9995	.9997	.9998	.9998	.9999	.9999	1.0000−

You can also use the standard normal table to find probabilities for *any* normal distribution by first converting values from the distribution to z-scores.

EXAMPLE 3 Use a z-score and the standard normal table

BIOLOGY Scientists conducted aerial surveys of a seal sanctuary and recorded the number x of seals they observed during each survey. The numbers of seals observed were normally distributed with a mean of 73 seals and a standard deviation of 14.1 seals. Find the probability that at most 50 seals were observed during a survey.

Solution

STEP 1 **Find** the z-score corresponding to an x-value of 50.

$$z = \frac{x - \bar{x}}{\sigma} = \frac{50 - 73}{14.1} \approx -1.6$$

STEP 2 **Use** the table to find $P(x \le 50) \approx P(z \le -1.6)$.

The table shows that $P(z \le -1.6) = 0.0548$. So, the probability that at most 50 seals were observed during a survey is about 0.0548.

z	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
−3	.0013	.0010	.0007	.0005	.0003	.0002	.0002	.0001	.0001	.0000+
−2	.0228	.0179	.0139	.0107	.0082	.0062	.0047	.0035	.0026	.0019
−1	.1587	.1357	.1151	.0968	.0808	.0668	.0548	.0446	.0359	.0287
−0	.5000	.4602	.4207	.3821	.3446	.3085	.2743	.2420	.2119	.1841
0	.5000	.5398	.5793	.6179	.6554	.6915	.7257	.7580	.7881	.8159

✓ GUIDED PRACTICE for Example 3

8. **WHAT IF?** In Example 3, find the probability that at most 90 seals were observed during a survey.

9. **REASONING** *Explain* why it makes sense that $P(z \le 0) = 0.5$.

6.3 EXERCISES

HOMEWORK KEY
○ = See **WORKED-OUT SOLUTIONS** Exs. 3, 11, and 33
★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 17, 18, 28, and 35

SKILL PRACTICE

1. **VOCABULARY** Copy and complete: A(n) __?__ is a bell-shaped curve that is symmetric about the mean.

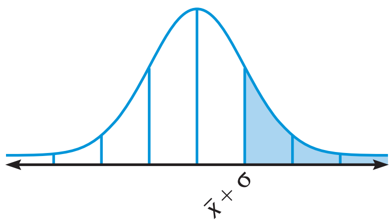
2. ★ **WRITING** *Describe* how to use the standard normal table to find $P(z \le 1.4)$.

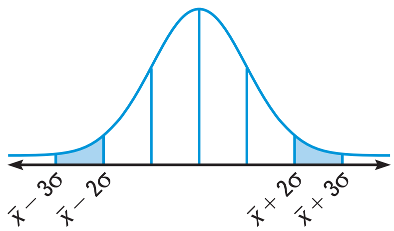
EXAMPLE 1 for Exs. 3–10

FIND A NORMAL PROBABILITY A normal distribution has mean $\overline{x}$ and standard deviation σ. Find the indicated probability for a randomly selected *x*-value from the distribution.

3. $P(x \le \overline{x} - \sigma)$
4. $P(x \ge \overline{x} + 2\sigma)$
5. $P(x \le \overline{x} + \sigma)$
6. $P(x \ge \overline{x} - \sigma)$
7. $P(\overline{x} - \sigma \le x \le \overline{x} + \sigma)$
8. $P(\overline{x} - 3\sigma \le x \le \overline{x})$

USING A NORMAL CURVE Give the percent of the area under the normal curve represented by the shaded region.

9. 

10. 

EXAMPLE 2 for Exs. 11–18

NORMAL DISTRIBUTIONS A normal distribution has a mean of 33 and a standard deviation of 4. Find the probability that a randomly selected *x*-value from the distribution is in the given interval.

11. Between 29 and 37
12. Between 33 and 45
13. Between 21 and 41
14. At least 25
15. At least 29
16. At most 37

17. ★ **MULTIPLE CHOICE** A normal distribution has a mean of 84 and a standard deviation of 5. What is the probability that a randomly selected *x*-value from the distribution is between 74 and 94?

Ⓐ 0.475 Ⓑ 0.68 Ⓒ 0.95 Ⓓ 0.997

18. ★ **MULTIPLE CHOICE** A normal distribution has a mean of 51 and a standard deviation of 3. What is the probability that a randomly selected *x*-value from the distribution is at most 48?

Ⓐ 0.025 Ⓑ 0.16 Ⓒ 0.84 Ⓓ 0.975

EXAMPLE 3 for Exs. 19–27

STANDARD NORMAL TABLE A normal distribution has a mean of 64 and a standard deviation of 7. Use the standard normal table to find the indicated probability for a randomly selected *x*-value from the distribution.

19. $P(x \le 68)$
20. $P(x \le 80)$
21. $P(x \le 45)$
22. $P(x \le 54)$
23. $P(x \le 64)$
24. $P(x \ge 59)$
25. $P(x \ge 75)$
26. $P(60 \le x \le 75)$
27. $P(45 \le x \le 65)$

28. ★ **WRITING** When n% of the data are less than or equal to a certain value, that value is called the nth percentile. For normally distributed data, describe the value that represents the 84th percentile in terms of the mean and standard deviation.

29. **ERROR ANALYSIS** In a study, the wheat yields (in bushels) for several plots of land were normally distributed with a mean of 4 bushels and a standard deviation of 0.25 bushel. *Describe* and correct the error in finding the probability that a plot yielded at least 3.8 bushels.

$$z = \frac{x - \bar{x}}{\sigma} = \frac{3.8 - 4}{0.25} = -0.8$$

From the standard normal table, $P(z \geq -0.8) = 0.2119$. So, the probability that a plot yielded at least 3.8 bushels is 0.2119.

30. **CHALLENGE** A normal curve is defined by an equation of this form:

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x - \bar{x}}{\sigma}\right)^2}$$

a. **Graphing Calculator** Graph three equations of the given form. The equations should use the same mean but different standard deviations.

b. **Reasoning** *Describe* the effect of the standard deviation on the shape of a normal curve.

PROBLEM SOLVING

EXAMPLES 2 and 3 for Exs. 31–34

31. **BIOLOGY** The illustration shows a housefly at several times its actual size and indicates the fly's wing length. A study found that the wing lengths of houseflies are normally distributed with a mean of about 4.6 millimeters and a standard deviation of about 0.4 millimeter. What is the probability that a randomly selected housefly has a wing length of at least 5 millimeters?

32. **FIRE DEPARTMENT** The time a fire department takes to arrive at the scene of an emergency is normally distributed with a mean of 6 minutes and a standard deviation of 1 minute.

a. What is the probability that the fire department takes at most 8 minutes to arrive at the scene of an emergency?

b. What is the probability that the fire department takes between 4 minutes and 7 minutes to arrive at the scene of an emergency?

33. **MULTI-STEP PROBLEM** Boxes of cereal are filled by a machine. Tests of the machine's accuracy show that the amount of cereal in each box varies. The weights are normally distributed with a mean of 20 ounces and a standard deviation of 0.25 ounce.

a. Find the z-scores for weights of 19.4 ounces and 20.4 ounces.

b. What is the probability that a randomly selected cereal box weighs at most 19.4 ounces?

c. What is the probability that a randomly selected cereal box weighs between 19.4 ounces and 20.4 ounces? *Explain* your reasoning.

34. BOTANY The guayule plant, which grows in the southwestern United States and in Mexico, is one of several plants that can be used as a source of rubber. In a large group of guayule plants, the heights of the plants are normally distributed with a mean of 12 inches and a standard deviation of 2 inches.

a. What percent of the plants are taller than 16 inches?

b. What percent of the plants are at most 13 inches?

c. What percent of the plants are between 7 inches and 14 inches?

d. What percent of the plants are at least 3 inches taller than or at least 3 inches shorter than the mean height?

Guayule plants

35. ★ EXTENDED RESPONSE Lisa and Ann took different college entrance tests. The scores on the test that Lisa took are normally distributed with a mean of 20 points and a standard deviation of 4.2 points. The scores on the test that Ann took are normally distributed with a mean of 500 points and a standard deviation of 90 points. Lisa scored 30 on her test, and Ann scored 610 on her test.

a. Calculate Find the z-score for Lisa's test score.

b. Calculate Find the z-score for Ann's test score.

c. Interpret Which student scored better on her college entrance test? *Explain* your reasoning.

36. CHALLENGE According to a survey by the National Center for Health Statistics, the heights of adult men in the United States are normally distributed with a mean of 69 inches and a standard deviation of 2.75 inches.

a. If you randomly choose 3 adult men, what is the probability that all of them are more than 6 feet tall?

b. What is the probability that 5 randomly selected men all have heights between 65 inches and 75 inches?

Find the Area Under a Normal Curve

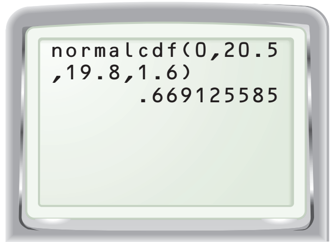
MATHEMATICAL PRACTICES
Use appropriate tools strategically.

QUESTION How can you use a graphing calculator to find the area under a normal curve?

EXAMPLE 1 Find a normal probability

The lengths of a group of newborn babies are normally distributed with a mean of 19.8 inches and a standard deviation of 1.6 inches. Find the probability that a randomly chosen baby in this group is at most 20.5 inches long.

STEP 1 ***Clarify the problem*** We want to find $P(x \le 20.5)$ for a normal distribution with a mean of 19.8 and a standard deviation of 1.6.



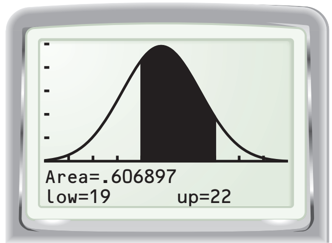
STEP 2 ***Enter information*** Access the "Distributions" menu by pressing 2nd [VARS]. Select "normalcdf(," then enter (lower bound, upper bound, mean, standard deviation). Use 0 as the lower bound. You can see at the right that the probability is about 66.9%.

EXAMPLE 2 Find and display an area under a normal curve

Find $P(19 \le x \le 22)$ for the situation in Example 1.

STEP 1 ***Enter information*** In the Distributions menu, select "DRAW" at the top and then "ShadeNorm(." You must enter (lower bound, upper bound, mean, standard deviation). Enter 19 for the lower bound and 22 for the upper bound. Do not yet press ENTER.

STEP 2 ***Choose a viewing window*** For a window that shows about 3 standard deviations to either side of the mean, let Xmin = 15 and Xmax = 25. The area under the curve is 1, so the maximum Y-value will be well less than 1. Let Ymin = −0.075 and Ymax = 0.25. Return to the home screen and press ENTER. You can see at the right that the area is about 0.607, so the probability is about 60.7%.



PRACTICE

1. The mean number of potatoes per 15 pound bag from a supplier is 42, with a standard deviation of 4.5. The distribution is normal. Find the probability that a randomly selected bag contains the given number of potatoes.

   a. 50 or fewer **b.** at least 35 **c.** 37 to 47 **d.** 40 to 44

2. Scores on a qualifying exam are normally distributed with a mean score of 230 and a standard deviation of 43. A score of 250 is required to pass.

   a. Find and display the area under the normal curve for passing scores.

   b. Find and display the area under the normal curve for scores of 200–250. (*Note:* You will first need to clear the drawing from part (a).)