LESSON 6

Counting Techniques

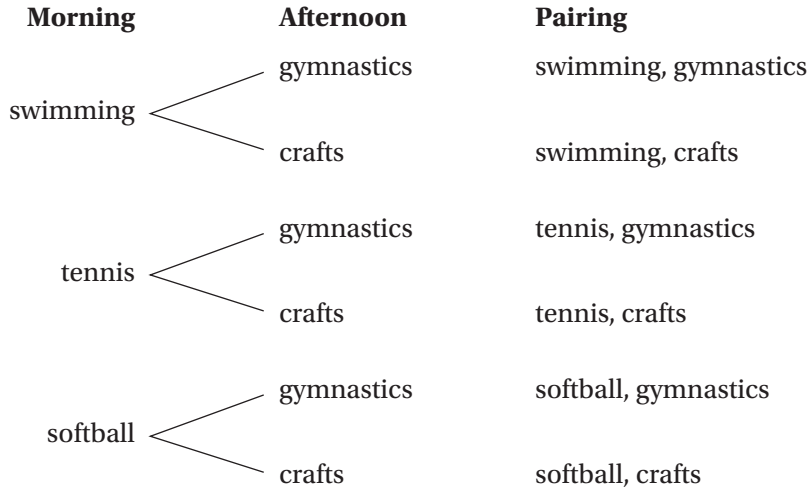
GOAL Count outcomes using tree diagrams, the counting principle, and formulas for permutations and combinations.

When you perform an experiment, the possible results are called **outcomes**. An **event** is a collection of outcomes. For example, one possible outcome of flipping two coins is "heads, tails" and an event is "at least one coin shows tails." A **tree diagram** can help you find the possible outcomes of an experiment by using branching to list choices.

EXAMPLE 1 Making a Tree Diagram

A summer camp offers a choice of swimming, tennis, or softball in the morning and gymnastics or crafts in the afternoon. How many different pairings of one morning activity and one afternoon activity are possible?

SOLUTION



There are six different pairings of morning and afternoon activities.

Activity

Determining Outcomes

1. Suppose a number cube is rolled and a coin is flipped. Copy and complete the table to find the possible outcomes.
2. How many possible outcomes are listed in your completed table?
3. Multiply the number of possible outcomes of rolling a number cube and the number of possible outcomes of flipping a coin. What do you notice?

	H	T
1	1, H	1, T
2	2, H	2, T
3	?	?
4	?	?
5	?	?
6	?	?

The *counting principle*, which is shown below, allows you to use multiplication to find the number of outcomes of two or more events.

The Counting Principle

If one event can occur in m ways, and for each of these a second event can occur in n ways, then the number of ways that the two events can occur together is $m \cdot n$.

The counting principle can be extended to three or more events.

EXAMPLE 2 Using the Counting Principle

In one state, the standard configuration of a license plate is 3 letters followed by 4 digits, as shown below. How many different license plates can be made if the letters and the digits can be repeated?

SOLUTION

There are 26 choices for each letter and 10 choices for each digit. Use the counting principle to find the number of different license plates that can be made.

$$\text{Number of license plates} = 26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10$$

$$= 175{,}760{,}000$$

There are 175,760,000 different license plates that can be made.

CHECK Examples 1 and 2

Make a tree diagram to find the number of outcomes.

1. At Jake's Pizza, the pizza of the day comes with a choice of crust (thin or thick), one meat topping (sausage, pepperoni, or ham), and one vegetable topping (onions, mushrooms, olives, or peppers). In how many ways can you order a pizza of the day?
2. A coin is flipped 3 times. How many outcomes are possible?

Use the counting principle to find the number of outcomes.

3. In your class elections, Ty and Sue are running for president, Frank and Li are running for vice-president, Andrew, Jerry, and Paige are running for treasurer, and Matt and Keesha are running for secretary. How many different election results are possible?
4. A number cube is rolled 4 times. How many outcomes are possible?

Permutations A **permutation** is an arrangement of a group of objects in a particular order. For example, there are six permutations of the letters X, Y, and Z:

XYZ, XZY, YXZ, YZX, ZXY, and ZYX

The counting principle can be used to determine the number of permutations of the letters X, Y, and Z, as shown below.

Choices for first letter	·	Choices for second letter	·	Choices for second letter	=	Ways to arrange the letters
3	·	2	·	1	=	6

EXAMPLE 3 Finding the Number of Permutations

There are 18 school bands participating in a competition. In how many ways can first, second, and third places be awarded?

SOLUTION

There are 18 choices for first place, 17 choices for second place, and 16 choices for third place. Use the counting principle to find the number of ways that first, second, and third places can be awarded.

Choices for first place	·	Choices for second place	·	Choices for third place	=	Ways to award places
18	·	17	·	16	=	4896

There are 4896 ways that first, second, and third places can be awarded.

The formulas below can also be used to determine the number of permutations. These formulas include *factorial* numbers. The expression $n!$ is read as "n factorial" and represents the product of all integers from 1 to n. (0! is defined to be equal to 1.)

Formulas for Permutations

The number of permutations of n distinct objects is denoted by ${}_nP_n$ and is given by:

$$\begin{aligned} {}_nP_n &= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \\ &= n! \end{aligned}$$

The number of permutations of r objects taken from a group of n distinct objects is denoted by ${}_nP_r$ and is given by:

$$\begin{aligned} {}_nP_r &= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

EXAMPLE 4 Using a Formula for Permutations

There are 9 tourist attractions that you want to visit during your vacation. You don't have enough time to visit all 9, so you decide to visit only 4 of the attractions. In how many orders can you visit 4 of the attractions?

SOLUTION

Find the number of permutations of 9 objects taken 4 at a time.

$$_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 3024$$

There are 3024 orders that you can visit 4 of the attractions.

Combinations In a permutation, the order of the objects is important. A **combination** is an arrangement of a group of objects where the order is *not* important. For example, suppose you are dealt 7 cards at the beginning of a card game. The order in which your cards are dealt is not important.

Formula For Combinations

The number of combinations of r objects taken from a group of n distinct objects is denoted by $_nC_r$ and is given by:

$$_nC_r = \frac{n!}{(n-r)! \cdot r!}$$

EXAMPLE 5 Using the Formula for Combinations

The bookstore where you work is planning to have an "employee favorites" display, in which each employee displays 3 of his or her favorite books. To help you decide which 3 books to choose, you make a list of your favorite books. If there are 12 books on the list, in how many ways can you choose 3 books?

SOLUTION

Find the number of ways to choose 3 books from a group of 12 books.

$$_{12}C_3 = \frac{12!}{9! \cdot 3!} \qquad \textit{Use formula for combinations.}$$

$$= \frac{12 \cdot 11 \cdot 10 \cdot \cancel{9!}}{\cancel{9!} \cdot 3!} \qquad \textit{Divide out common factorial.}$$

$$= \frac{1320}{6} \qquad \textit{Multiply.}$$

$$= 220 \qquad \textit{Divide.}$$

There are 220 ways to choose the books.

CHECK **Examples 3, 4, and 5**

Find the number of permutations.

5. A basketball team has 12 players. In how many ways can the players line up to enter the gym at the beginning of the game?
6. How many 3 letter arrangements of the letters in the word RANDOMLY are possible if no letter may be repeated?

Find the number of combinations.

7. In how many ways can you choose 4 types of fish from the 26 types of fish at a pet store?
8. In how many ways can you choose 2 DVDs from the 31 DVDs in the new releases section at a movie rental store?

EXERCISES

Make a tree diagram to find the number of outcomes.

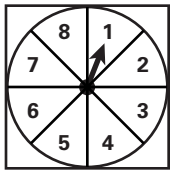
1. You are ordering practice T-shirts for your soccer team. You can get the T-shirts in one of the three school colors (blue, gold, or white) and you can get long-sleeve or short-sleeve T-shirts. How many T-shirts are possible?
2. You pack a solid shirt, a striped shirt, a plaid shirt, tan pants, and jeans for a weekend trip. How many outfits are possible?
3. At a restaurant, you have your choice of one entrée (beef, chicken, or vegetable stir-fry) and one side dish (baked potato, French fries, rice, or pasta). How many dinners are possible?
4. Jar A contains 3 marbles (one blue, one red, and one green), jar B contains 3 marbles (one black, one purple, and one yellow), and jar C contains 2 marbles (one orange and one white). How many outcomes are possible if you choose one marble from each jar?

Use the counting principle to find the number of outcomes.

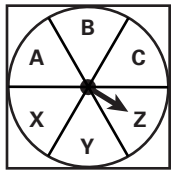
5. In one state, the standard configuration of a license plate is 3 letters followed by 3 digits. How many license plates can be made if the letters and the digits can be repeated?

6. You are given a random 4 digit personal identification number (PIN) for your bank card. How many 4 digit PINs are possible if the numbers can be repeated?
7. You roll a number cube 5 times. How many outcomes are possible?
8. Radio station *call letters,* such as WROC and KJAZ, consist of 4 letters. The call letters need to begin with either a W or a K. How many different call letters are possible if the letters can be repeated?

In Exercises 9 and 10, use the spinners shown.



Spinner 1



Spinner 2

9. Suppose you spin spinner 1, and then spin spinner 2. Find the number of outcomes.

10. Suppose you spin spinner 1, and then spin spinner 1 again. Find the number of outcomes.

Find the value of the expression.

11. ${}_7P_3$ **12.** ${}_{12}P_6$ **13.** ${}_9C_7$ **14.** ${}_{20}C_5$

Find the number of permutations.

15. Find the number of permutations of the letters in the word GUITAR.

16. A band has recorded 10 songs. In how many ways can they order the songs on their CD?

17. There are 16 divers participating in a diving competition. In how many ways can the divers be awarded first, second, and third places?

18. The combinations for the lockers at a school have 3 numbers. Each of the numbers is a number from 0 through 49. How many locker combinations are possible if the numbers cannot be repeated?

Find the number of combinations.

19. A cheerleading squad has 14 members. In how many ways can 2 co-captains be chosen?

20. In how many ways can you choose 3 different toppings for your ice cream from 10 toppings?

21. You want to buy 5 different shades of nail polish. In how many ways can you choose 5 shades from 42 shades?

22. In how many ways can a teacher choose a group of 6 students from a class of 26 students?

Tell whether the situation describes a *permutation* or *combination*. Then answer the question.

23. A group of 7 friends goes to a movie. In how many ways can they sit in a row that has 7 empty seats?

24. In how many ways can you choose 3 posters from the 24 posters at a music store?

25. You need to buy fruit for a fruit salad. In how many ways can you choose 6 different types of fruit from the 18 types of fruit at a grocery store?

26. In how many ways can the coach arrange the batting order of the 9 starting players on a baseball team?