7.3 Define and Use Zero and Negative Exponents

Before You used properties of exponents to simplify expressions.

Now You will use zero and negative exponents.

Why? So you can compare masses, as in Ex. 52.

Key Vocabulary
- **reciprocal**

In the activity, you saw what happens when you raise a number to a zero or negative exponent. The activity suggests the following definitions.

KEY CONCEPT *For Your Notebook*

Definition of Zero and Negative Exponents

Words	Algebra	Example
a to the zero power is 1.	$a^0 = 1,\ a \neq 0$	$5^0 = 1$
a^{-n} is the reciprocal of a^n.	$a^{-n} = \frac{1}{a^n},\ a \neq 0$	$2^{-1} = \frac{1}{2}$
a^n is the reciprocal of a^{-n}.	$a^n = \frac{1}{a^{-n}},\ a \neq 0$	$2 = \frac{1}{2^{-1}}$

COMMON CORE

CC.9-12.A.SSE.3c Use the properties of exponents to transform expressions for exponential functions.

EXAMPLE 1 Use definition of zero and negative exponents

SIMPLIFY EXPRESSIONS

In this lesson, when simplifying powers with numerical bases, evaluate the numerical power.

a. $3^{-2} = \frac{1}{3^2}$ — **Definition of negative exponents**

$= \frac{1}{9}$ — **Evaluate power.**

b. $(-7)^0 = 1$ — **Definition of zero exponent**

c. $\left(\frac{1}{5}\right)^{-2} = \frac{1}{\left(\frac{1}{5}\right)^2}$ — **Definition of negative exponents**

$= \frac{1}{\frac{1}{25}}$ — **Evaluate power.**

$= 25$ — **Simplify by multiplying numerator and denominator by 25.**

d. $0^{-5} = \cancel{\frac{1}{0^5}}$ (Undefined) — a^{-n} **is defined only for a *nonzero* number *a*.**

✓ **GUIDED PRACTICE** for Example 1

Evaluate the expression.

1. $\left(\frac{2}{3}\right)^0$ **2.** $(-8)^{-2}$ **3.** $\frac{1}{2^{-3}}$ **4.** $(-1)^0$

PROPERTIES OF EXPONENTS The properties of positive exponents you have already learned can be used with negative or zero exponents.

KEY CONCEPT *For Your Notebook*

Properties of Exponents

Let a and b be real numbers, and let m and n be integers.

$a^m \cdot a^n = a^{m+n}$	**Product of powers property**
$(a^m)^n = a^{mn}$	**Power of a power property**
$(ab)^m = a^m b^m$	**Power of a product property**
$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$	**Quotient of powers property**
$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$	**Power of a quotient property**

EXAMPLE 2 Evaluate exponential expressions

a. $6^{-4} \cdot 6^4 = 6^{-4+4}$ **Product of powers property**

$= 6^0$ **Add exponents.**

$= 1$ **Definition of zero exponent**

b. $(4^{-2})^2 = 4^{-2 \cdot 2}$ **Power of a power property**

$= 4^{-4}$ **Multiply exponents.**

$= \frac{1}{4^4}$ **Definition of negative exponents**

$= \frac{1}{256}$ **Evaluate power.**

c. $\frac{1}{3^{-4}} = 3^4$ **Definition of negative exponents**

$= 81$ **Evaluate power.**

d. $\frac{5^{-1}}{5^2} = 5^{-1-2}$ **Quotient of powers property**

$= 5^{-3}$ **Subtract exponents.**

$= \frac{1}{5^3}$ **Definition of negative exponents**

$= \frac{1}{125}$ **Evaluate power.**

GUIDED PRACTICE for Example 2

Evaluate the expression.

5. $\frac{1}{4^{-3}}$ **6.** $(5^{-3})^{-1}$ **7.** $(-3)^5 \cdot (-3)^{-5}$ **8.** $\frac{6^{-2}}{6^2}$

EXAMPLE 3 Use properties of exponents

Simplify the expression. Write your answer using only positive exponents.

a. $(2xy^{-5})^3 = 2^3 \cdot x^3 \cdot (y^{-5})^3$ — Power of a product property

$= 8 \cdot x^3 \cdot y^{-15}$ — Power of a power property

$= \dfrac{8x^3}{y^{15}}$ — Definition of negative exponents

b. $\dfrac{(2x)^{-2}y^5}{-4x^2y^2} = \dfrac{y^5}{(2x)^2(-4x^2y^2)}$ — Definition of negative exponents

$= \dfrac{y^5}{(4x^2)(-4x^2y^2)}$ — Power of a product property

$= \dfrac{y^5}{-16x^4y^2}$ — Product of powers property

$= -\dfrac{y^3}{16x^4}$ — Quotient of powers property

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★ EXAMPLE 4 Standardized Test Practice

The order of magnitude of the mass of a polyphemus moth larva when it hatches is 10^{-3} gram. During the first 56 days of its life, the moth larva can eat about 10^5 times its own mass in food. About how many grams of food can the moth larva eat during its first 56 days?

Ⓐ 10^{-15} gram Ⓑ 0.00000001 gram

Ⓒ 100 grams Ⓓ 10,000,000 grams

Not to scale

Solution

To find the amount of food the moth larva can eat in the first 56 days of its life, multiply its original mass, 10^{-3}, by 10^5.

$10^5 \cdot 10^{-3} = 10^{5+(-3)} = 10^2 = 100$

The moth larva can eat about 100 grams of food in the first 56 days of its life.

▶ The correct answer is C. Ⓐ Ⓑ Ⓒ Ⓓ

✓ GUIDED PRACTICE for Examples 3 and 4

9. Simplify the expression $\dfrac{3xy^{-3}}{9x^3y}$. Write your answer using only positive exponents.

10. **SCIENCE** The order of magnitude of the mass of a proton is 10^4 times greater than the order of magnitude of the mass of an electron, which is 10^{-27} gram. Find the order of magnitude of the mass of a proton.

7.3 EXERCISES

HOMEWORK KEY

○ = See **WORKED-OUT SOLUTIONS** Exs. 11 and 53

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 44, 45, 54, and 57

◆ = **MULTIPLE REPRESENTATIONS** Ex. 55

SKILL PRACTICE

1. **VOCABULARY** Which definitions or properties would you use to simplify the expression $3^5 \cdot 3^{-5}$? *Explain.*

2. ★ **WRITING** *Explain* why the expression 0^{-4} is undefined.

EXAMPLE 1
for Exs. 3–14

EVALUATING EXPRESSIONS **Evaluate the expression.**

3. 4^{-3} **4.** 7^{-3} **5.** $(-3)^{-1}$ **6.** $(-2)^{-6}$

7. 2^0 **8.** $(-4)^0$ **9.** $\left(\frac{3}{4}\right)^0$ **10.** $\left(\frac{-9}{16}\right)^0$

11. $\left(\frac{2}{7}\right)^{-2}$ **12.** $\left(\frac{4}{3}\right)^{-3}$ **13.** 0^{-3} **14.** 0^{-2}

EXAMPLE 2
for Exs. 15–27

15. $2^{-2} \cdot 2^{-3}$ **16.** $7^{-6} \cdot 7^4$ **17.** $(2^{-1})^5$ **18.** $(3^{-2})^2$

19. $\frac{1}{3^{-3}}$ **20.** $\frac{1}{6^{-2}}$ **21.** $\frac{3^{-3}}{3^2}$ **22.** $\frac{6^{-3}}{6^{-5}}$

23. $4\left(\frac{3}{2}\right)^{-1}$ **24.** $16\left(\frac{2^{-3}}{2^2}\right)$ **25.** $6^0 \cdot \left(\frac{1}{4^{-2}}\right)$ **26.** $3^{-2} \cdot \left(\frac{5}{7^0}\right)$

27. **ERROR ANALYSIS** *Describe* and correct the error in evaluating the expression $-6 \cdot 3^0$.

$$-6 \cdot 3^0 = -6 \cdot 0$$
$$= 0$$

EXAMPLE 3
for Exs. 28–43

SIMPLIFYING EXPRESSIONS **Simplify the expression. Write your answer using only positive exponents.**

28. x^{-4} **29.** $2y^{-3}$ **30.** $(4g)^{-3}$ **31.** $(-11h)^{-2}$

32. x^2y^{-3} **33.** $5m^{-3}n^{-4}$ **34.** $(6x^{-2}y^3)^{-3}$ **35.** $(-15fg^2)^0$

36. $\frac{r^{-2}}{s^{-4}}$ **37.** $\frac{x^{-5}}{y^2}$ **38.** $\frac{1}{8x^{-2}y^{-6}}$ **39.** $\frac{1}{15x^{10}y^{-8}}$

40. $\frac{1}{(-2z)^{-2}}$ **41.** $\frac{9}{(3d)^{-3}}$ **42.** $\frac{(3x)^{-3}y^4}{-x^2y^{-6}}$ **43.** $\frac{12x^8y^{-7}}{(4x^{-2}y^{-6})^2}$

44. ★ **MULTIPLE CHOICE** Which expression simplifies to $2x^4$?

Ⓐ $2x^{-4}$ Ⓑ $\frac{32}{(2x)^{-4}}$ Ⓒ $\frac{1}{2x^{-4}}$ Ⓓ $\frac{8}{4x^{-4}}$

45. ★ **MULTIPLE CHOICE** Which expression is equivalent to $(-4 \cdot 2^0 \cdot 3)^{-2}$?

Ⓐ -12 Ⓑ $-\frac{1}{144}$ Ⓒ 0 Ⓓ $\frac{1}{144}$

CHALLENGE **In Exercises 46–48, tell whether the statement is true for all nonzero values of a and b. If it is not true, give a counterexample.**

46. $\frac{a^{-3}}{a^{-4}} = \frac{1}{a}$

47. $\frac{a^{-1}}{b^{-1}} = \frac{b}{a}$

48. $a^{-1} + b^{-1} = \frac{1}{a + b}$

49. **CHALLENGE** Compare the values of a^n and a^{-n} when $n < 0$, when $n = 0$, and when $n > 0$ for **(a)** $a > 1$ and **(b)** $0 < a < 1$. *Explain* your reasoning.

PROBLEM SOLVING

EXAMPLE 4 for Exs. 50–54

50. **MASS** The mass of a grain of salt is about 10^{-4} gram. About how many grains of salt are in a box containing 100 grams of salt?

51. **MASS** The mass of a grain of a certain type of rice is about 10^{-2} gram. About how many grains of rice are in a box containing 10^3 grams of rice?

52. **BOTANY** The average mass of the fruit of the wolffia angusta plant is about 10^{-4} gram. The largest pumpkin ever recorded had a mass of about 10^4 kilograms. About how many times greater is the mass of the largest pumpkin than the mass of the fruit of the wolffia angusta plant?

53. **MEDICINE** A doctor collected about 10^{-2} liter of blood from a patient to run some tests. The doctor determined that a drop of the patient's blood, or about 10^{-6} liter, contained about 10^7 red blood cells. How many red blood cells did the entire sample contain?

54. ★ **SHORT RESPONSE** One of the smallest plant seeds comes from an orchid, and one of the largest plant seeds comes from a giant fan palm. A seed from an orchid has a mass of 10^{-9} gram and is 10^{13} times less massive than a seed from a giant fan palm. A student says that the seed from the giant fan palm has a mass of about 1 kilogram. Is the student correct? *Explain*.

Orchid

Giant fan palm

55. **MULTIPLE REPRESENTATIONS** Consider folding a piece of paper in half a number of times.

a. **Making a Table** Each time the paper is folded, record the number of folds and the fraction of the original area in a table like the one shown.

Number of folds	0	1	2	3
Fraction of original area	?	?	?	?

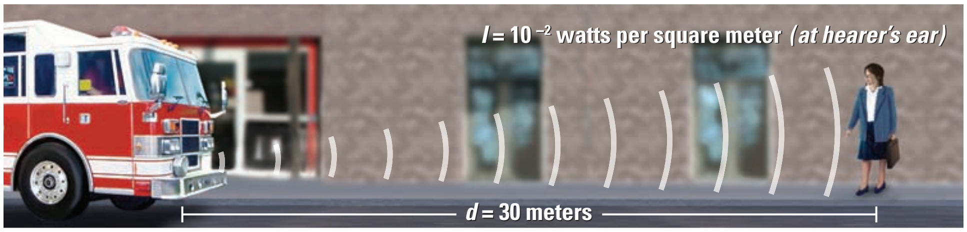
b. **Writing an Expression** Write an exponential expression for the fraction of the original area of the paper using a base of $\frac{1}{2}$.

56. SCIENCE Diffusion is the movement of molecules from one location to another. The time t (in seconds) it takes molecules to diffuse a distance of x centimeters is given by $t = \frac{x^2}{2D}$ where D is the diffusion coefficient.

a. You can examine a cross section of a drop of ink in water to see how the ink diffuses. The diffusion coefficient for the molecules in the drop of ink is about 10^{-5} square centimeter per second. How long will it take the ink to diffuse 1 micrometer (10^{-4} centimeter)?

b. Check your answer to part (a) using unit analysis.

57. ★ EXTENDED RESPONSE The intensity of sound I (in watts per square meter) can be modeled by $I = 0.08Pd^{-2}$ where P is the power (in watts) of the sound's source and d is the distance (in meters) that you are from the source of the sound.



Not to scale

a. What is the power (in watts) of the siren of the firetruck shown in the diagram?

b. Using the power of the siren you found in part (a), simplify the formula for the intensity of sound from the siren.

c. *Explain* what happens to the intensity of the siren when you double your distance from it.

58. CHALLENGE Coal can be burned to generate energy. The heat energy in 1 pound of coal is about 10^4 BTU (British Thermal Units). Suppose you have a stereo. It takes about 10 pounds of coal to create the energy needed to power the stereo for 1 year.

a. About how many BTUs does your stereo use in 1 year?

b. Suppose the power plant that delivers energy to your home produces 10^{-1} pound of sulfur dioxide for each 10^6 BTU of energy that it creates. How much sulfur dioxide is added to the air by generating the energy needed to power your stereo for 1 year?

Extension Define and Use Fractional Exponents

GOAL Use fractional exponents.

Key Vocabulary
- cube root

COMMON CORE

CC.9-12.N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents.

You have learned to write the square root of a number using a radical sign. You can also write a square root of a number using exponents.

For any $a \geq 0$, suppose you want to write $\sqrt{a}$ as a^k. Recall that a number b (in this case, a^k) is a square root of a number a provided $b^2 = a$. Use this definition to find a value for k as follows.

$$b^2 = a \qquad \text{Definition of square root}$$

$$(a^k)^2 = a \qquad \text{Substitute } a^k \text{ for } b.$$

$$a^{2k} = a^1 \qquad \text{Power of a power property}$$

Because the bases are the same in the equation $a^{2k} = a^1$, the exponents must be equal:

$$2k = 1 \qquad \text{Set exponents equal.}$$

$$k = \frac{1}{2} \qquad \text{Solve for } k.$$

So, for a nonnegative number a, $\sqrt{a} = a^{1/2}$.

You can work with exponents of $\frac{1}{2}$ and multiples of $\frac{1}{2}$ just as you work with integer exponents.

EXAMPLE 1 Evaluate expressions involving square roots

a. $16^{1/2} = \sqrt{16}$
$= 4$

b. $25^{-1/2} = \dfrac{1}{25^{1/2}}$
$= \dfrac{1}{\sqrt{25}}$
$= \dfrac{1}{5}$

c. $9^{5/2} = 9^{(1/2) \cdot 5}$
$= \left(9^{1/2}\right)^5$
$= \left(\sqrt{9}\right)^5$
$= 3^5$
$= 243$

d. $4^{-3/2} = 4^{(1/2) \cdot (-3)}$
$= \left(4^{1/2}\right)^{-3}$
$= \left(\sqrt{4}\right)^{-3}$
$= 2^{-3}$
$= \dfrac{1}{2^3}$
$= \dfrac{1}{8}$

FRACTIONAL EXPONENTS You can work with other fractional exponents just as you did with $\frac{1}{2}$.

CUBE ROOTS If $b^3 = a$, then b is the **cube root** of a. For example, $2^3 = 8$, so 2 is the cube root of 8. The cube root of a can be written as $\sqrt[3]{a}$ or $a^{1/3}$.

EXAMPLE 2 Evaluate expressions involving cube roots

a. $27^{1/3} = \sqrt[3]{27}$
$= \sqrt[3]{3^3}$
$= 3$

b. $8^{-1/3} = \frac{1}{8^{1/3}}$
$= \frac{1}{\sqrt[3]{8}}$
$= \frac{1}{2}$

c. $64^{4/3} = 64^{(1/3) \cdot 4}$
$= (64^{1/3})^4$
$= (\sqrt[3]{64})^4$
$= 4^4$
$= 256$

d. $125^{-2/3} = 125^{(1/3) \cdot (-2)}$
$= (125^{1/3})^{-2}$
$= (\sqrt[3]{125})^{-2}$
$= 5^{-2}$
$= \frac{1}{5^2}$
$= \frac{1}{25}$

PROPERTIES OF EXPONENTS The properties of exponents for integer exponents also apply to fractional exponents.

EXAMPLE 3 Use properties of exponents

a. $12^{-1/2} \cdot 12^{5/2} = 12^{(-1/2) + (5/2)}$
$= 12^{4/2}$
$= 12^2$
$= 144$

b. $\frac{6^{4/3} \cdot 6}{6^{1/3}} = \frac{6^{(4/3) + 1}}{6^{1/3}}$
$= \frac{6^{7/3}}{6^{1/3}}$
$= 6^{(7/3) - (1/3)}$
$= 6^2$
$= 36$

PRACTICE

EXAMPLES 1, 2, and 3 for Exs. 1–12

Evaluate the expression.

1. $100^{3/2}$
2. $121^{-1/2}$
3. $81^{-3/2}$
4. $216^{2/3}$
5. $27^{-1/3}$
6. $343^{-2/3}$
7. $9^{7/2} \cdot 9^{-3/2}$
8. $\left(\frac{1}{16}\right)^{1/2}\left(\frac{1}{16}\right)^{-1/2}$
9. $36^{5/2} \cdot \frac{36^{-1/2}}{(36^{-1})^{-7/2}}$
10. $(27^{-1/3})^3$
11. $(-64)^{-5/3}(-64)^{4/3}$
12. $(-8)^{1/3}(-8)^{-2/3}(-8)^{1/3}$
13. **REASONING** Let $x > 0$. Compare the values of $x^{1/2}$ and $x^{-1/2}$. Give examples to support your thinking.

MIXED REVIEW of Problem Solving

MATHEMATICAL PRACTICES Make sense of problems and persevere in solving them.

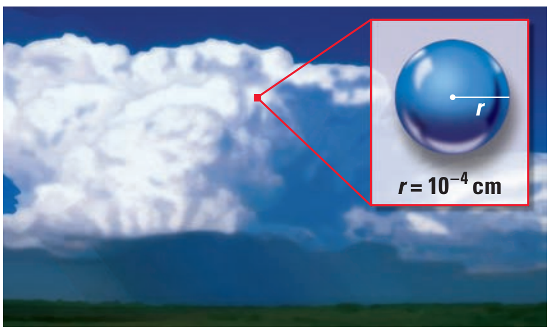
1. **GRIDDED ANSWER** In 2004 the fastest computers could record about 10^9 bits per second. (A bit is the smallest unit of memory storage for computers.) Scientists believed that the speed limit at the time was about 10^{12} bits per second. About how many times more bits per second was the speed limit than the fastest computers?

2. **MULTI-STEP PROBLEM** An office supply store sells cubical containers that can be used to store paper clips, rubber bands, or other supplies.

   a. One of the containers has a side length of $4\frac{1}{2}$ inches. Find the container's volume by writing the side length as an improper fraction and substituting the length into the formula for the volume of a cube.

   b. Identify the property of exponents you used to find the volume in part (a).

3. **SHORT RESPONSE** Clouds contain millions of tiny spherical water droplets. The radius of one droplet is shown.



   a. Find the order of magnitude of the volume of the droplet.

   b. Droplets combine to form raindrops. The radius of a raindrop is about 10^2 times greater than the droplet's radius. Find the order of magnitude of the volume of the raindrop.

   c. *Explain* how you can find the number of droplets that combine to form the raindrop. Then find the number of droplets and identify any properties of exponents you used.

4. **GRIDDED ANSWER** The least intense sound that is audible to the human ear has an intensity of about 10^{-12} watt per square meter. The intensity of sound from a jet engine at a distance of 30 meters is about 10^{15} times greater than the least intense sound. Find the intensity of sound from the jet engine.

5. **EXTENDED RESPONSE** For an experiment, a scientist dropped a spoonful, or about 10^{-1} cubic inch, of biodegradable olive oil into a pond to see how the oil would spread out over the surface of the pond. The scientist found that the oil spread until it covered an area of about 10^5 square inches.

   a. About how thick was the layer of oil that spread out across the pond? Check your answer using unit analysis.

   b. The pond has a surface area of 10^7 square inches. If the oil spreads to the same thickness as in part (a), how many cubic inches of olive oil would be needed to cover the entire surface of the pond?

   c. *Explain* how you could find the amount of oil needed to cover a pond with a surface area of 10^x square inches.

6. **OPEN-ENDED** The table shows units of measurement of time and the durations of the units in seconds.

Name of unit	Duration (seconds)
Gigasecond	10^9
Megasecond	10^6
Millisecond	10^{-3}
Nanosecond	10^{-9}

   a. Use the table to write a conversion problem that can be solved by applying a property of exponents involving products.

   b. Use the table to write a conversion problem that can be solved by applying a property of exponents involving quotients.