7.4 Write and Graph Exponential Growth Functions

Before	You wrote and graphed linear models.
Now	You will write and graph exponential growth models.
Why?	So you can find the value of a collector car, as in Example 4.

Key Vocabulary
- exponential function
- exponential growth
- compound interest

An **exponential function** is a function of the form $y = ab^x$ where $a \neq 0$, $b > 0$, and $b \neq 1$. Exponential functions are *nonlinear* functions. Observe how an exponential function compares with a linear function.

Linear function: $y = 3x + 2$

		+1	+1	+1	+1	
x	−2	−1	**0**	1	2
y	−4	−1	**2**	5	8
	+3	+3	+3	+3	

Exponential function: $y = 2 \cdot 3^x$

		+1	+1	+1	+1	
x	−2	−1	**0**	1	2
y	$\frac{2}{9}$	$\frac{2}{3}$	**2**	6	18
	×3	×3	×3	×3	

EXAMPLE 1 Write a function rule

ANALYZE RATE OF CHANGE
Notice that for an exponential function, the rate of change in y with respect to x is not constant as it is for a linear function. For instance, $\frac{4 - 2}{-1 - (-2)} = 2$, while $\frac{8 - 4}{0 - (-1)} = 4$.

Write a rule for the function.

x	−2	−1	0	1	2
y	2	4	8	16	32

Solution

STEP 1 **Tell** whether the function is exponential.

		+1	+1	+1	+1	
x	−2	−1	0	1	2
y	2	4	8	16	32
	×2	×2	×2	×2	

Here, the y-values are multiplied by 2 for each increase of 1 in x, so the table represents an exponential function of the form $y = ab^x$ where $b = 2$.

STEP 2 **Find** the value of a by finding the value of y when $x = 0$. When $x = 0$, $y = ab^0 = a \cdot 1 = a$. The value of y when $x = 0$ is 8, so $a = 8$.

STEP 3 **Write** the function rule. A rule for the function is $y = 8 \cdot 2^x$.

✓ **GUIDED PRACTICE** for Example 1

1. Write a rule for the function.

x	−2	−1	0	1	2
y	3	9	27	81	243

EXAMPLE 2 Graph an exponential function

Graph the function $y = 2^x$. Identify its domain and range.

Solution

STEP 1 **Make** a table by choosing a few values for x and finding the values of y. The domain is all real numbers.

x	-2	-1	0	1	2
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
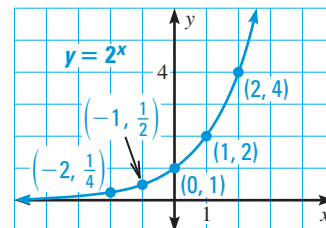
STEP 2 **Plot** the points.

STEP 3 **Draw** a smooth curve through the points. From either the table or the graph, you can see that the range is all positive real numbers.

EXAMPLE 3 Compare graphs of exponential functions

Graph the functions $y = 3 \cdot 2^x$ and $y = -3 \cdot 2^x$. Compare each graph with the graph of $y = 2^x$.
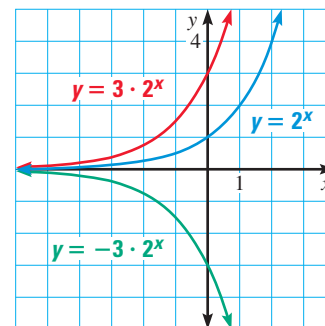
Solution

To graph each function, make a table of values, plot the points, and draw a smooth curve through the points.

x	$y = 2^x$	$y = 3 \cdot 2^x$	$y = -3 \cdot 2^x$
-2	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{3}{4}$
-1	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{3}{2}$
0	1	3	-3
1	2	6	-6
2	4	12	-12

Because the y-values for $y = 3 \cdot 2^x$ are 3 times the corresponding y-values for $y = 2^x$, the graph of $y = 3 \cdot 2^x$ is a vertical stretch of the graph of $y = 2^x$.

Because the y-values for $y = -3 \cdot 2^x$ are -3 times the corresponding y-values for $y = 2^x$, the graph of $y = -3 \cdot 2^x$ is a vertical stretch with a reflection in the x-axis of the graph of $y = 2^x$.

✓ **GUIDED PRACTICE** **for Examples 2 and 3**

2. Graph $y = 5^x$ and identify its domain and range.

3. Graph $y = \frac{1}{3} \cdot 2^x$. Compare the graph with the graph of $y = 2^x$.

4. Graph $y = -\frac{1}{3} \cdot 2^x$. Compare the graph with the graph of $y = 2^x$.

EXPONENTIAL GROWTH When $a > 0$ and $b > 1$, the function $y = ab^x$ represents **exponential growth**. When a quantity grows exponentially, it increases by the same percent over equal time periods. To find the amount to which the quantity grows after t time periods, use the following model.

KEY CONCEPT   *For Your Notebook*

Exponential Growth Model

a is the **initial amount**. r is the **growth rate**.

$$y = a(1 + r)^t$$

$1 + r$ is the **growth factor**. t is the **time period**.

REWRITE
EQUATIONS
Notice that you can rewrite $y = ab^x$ as $y = a(1 + r)^t$ by replacing b with $1 + r$ and x with t (for time).

Notice the relationship between the growth rate r and the growth factor $1 + r$. If the initial amount of a quantity is a units and the quantity is growing at a rate of r, then after one time period the new amount is:

Initial amount + amount of increase $= a + r \cdot a = a(1 + r)$

EXAMPLE 4   **Solve a multi-step problem**

ANOTHER WAY
For alternative methods for solving Example 4, see the **Problem Solving Workshop**.

COLLECTOR CAR The owner of a 1953 Hudson Hornet convertible sold the car at an auction. The owner bought it in 1984 when its value was $11,000. The value of the car increased at a rate of 6.9% per year.

a. Write a function that models the value of the car over time.

b. The auction took place in 2004. What was the approximate value of the car at the time of the auction? Round your answer to the nearest dollar.

Solution

a. Let C be the value of the car (in dollars), and let t be the time (in years) since 1984. The initial value a is $11,000, and the growth rate r is 0.069.

$C = a(1 + r)^t$ **Write exponential growth model.**

$= 11{,}000(1 + 0.069)^t$ **Substitute 11,000 for a and 0.069 for r.**

$= 11{,}000(1.069)^t$ **Simplify.**

AVOID ERRORS
The growth rate in this example is 6.9%, or 0.069. So, the growth factor is $1 + 0.069$, or 1.069, not 0.069.

b. To find the value of the car in 2004, 20 years after 1984, substitute 20 for t.

$C = 11{,}000(1.069)^{20}$ **Substitute 20 for t.**

$\approx 41{,}778$ **Use a calculator.**

▶ In 2004 the value of the car was about $41,778.

Animated Algebra at my.hrw.com

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COMPOUND INTEREST **Compound interest** is interest earned on both an initial investment and on previously earned interest. Compounding of interest can be modeled by exponential growth where a is the initial investment, r is the annual interest rate, and t is the number of years the money is invested.

⭐ **EXAMPLE 5** **Standardized Test Practice**

> **You put $250 in a savings account that earns 4% annual interest compounded yearly. You do not make any deposits or withdrawals. How much will your investment be worth in 5 years?**
>
> Ⓐ $300 Ⓑ $304.16 Ⓒ $1344.56 Ⓓ $781,250

ESTIMATE
You can use the simple interest formula, $I = prt$, to estimate the amount of interest earned: $(250)(0.04)(5) = 50$. Compounding interest will result in slightly more than $50.

Solution

$$y = a(1 + r)^t \qquad \text{Write exponential growth model.}$$
$$= 250(1 + 0.04)^5 \qquad \text{Substitute 250 for } a, 0.04 \text{ for } r, \text{ and 5 for } t.$$
$$= 250(1.04)^5 \qquad \text{Simplify.}$$
$$\approx 304.16 \qquad \text{Use a calculator.}$$

You will have $304.16 in 5 years.

▶ The correct answer is B. Ⓐ Ⓑ Ⓒ Ⓓ

✓ **GUIDED PRACTICE** **for Examples 4 and 5**

5. WHAT IF? In Example 4, suppose the owner of the car sold it in 1994. Find the value of the car to the nearest dollar.

6. WHAT IF? In Example 5, suppose the annual interest rate is 3.5%. How much will your investment be worth in 5 years?

7.4 EXERCISES

HOMEWORK KEY
◯ = See **WORKED-OUT SOLUTIONS**
Exs. 13 and 41
⭐ = **STANDARDIZED TEST PRACTICE**
Exs. 3, 8, 35, 42, 43, 46, and 50
◆ = **MULTIPLE REPRESENTATIONS**
Exs. 34, 41

SKILL PRACTICE

1. **VOCABULARY** In the exponential growth model $y = a(1 + r)^t$, the quantity $1 + r$ is called the __?__.

2. **VOCABULARY** For what values of b does the exponential function $y = ab^x$ (where $a > 0$) represent exponential growth?

3. ⭐ **WRITING** How does the graph of $y = 2 \cdot 5^x$ compare with the graph of $y = 5^x$? *Explain.*

EXAMPLE 1
for Exs. 4–8

WRITING FUNCTIONS **Write a rule for the function.**

4.

x	−2	−1	0	1	2
y	1	2	4	8	16

5.

x	−2	−1	0	1	2
y	5	25	125	625	3125

6.

x	−2	−1	0	1	2
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2

7.

x	−2	−1	0	1	2
y	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1

8. ★ WRITING Given a table of values, describe how can you tell if the table represents a linear function or an exponential function.

EXAMPLE 2
for Exs. 9–21

GRAPHING FUNCTIONS **Graph the function and identify its domain and range.**

9. $y = 4^x$

10. $y = 7^x$

11. $y = 8^x$

12. $y = 9^x$

13. $y = (1.5)^x$

14. $y = (2.5)^x$

15. $y = (1.2)^x$

16. $y = (4.3)^x$

17. $y = \left(\frac{4}{3}\right)^x$

18. $y = \left(\frac{7}{2}\right)^x$

19. $y = \left(\frac{5}{3}\right)^x$

20. $y = \left(\frac{5}{4}\right)^x$

21. ERROR ANALYSIS The price P (in dollars) of a pound of flour was $.27 in 1999. The price has increased by about 2% each year. Let t be the number of years since 1999. *Describe* and correct the error in finding the price of a pound of flour in 2002.

$P = a(1 + r)^t$

$= 0.27(1 + 2)^3 = 0.27(3)^3 = 7.29$

In 2002 the price of a pound of flour was $7.29.

EXAMPLE 3
for Exs. 22–34

COMPARING GRAPHS OF FUNCTIONS **Graph the function. Compare the graph with the graph of $y = 3^x$.**

22. $y = 2 \cdot 3^x$

23. $y = 4 \cdot 3^x$

24. $y = \frac{1}{4} \cdot 3^x$

25. $y = \frac{2}{3} \cdot 3^x$

26. $y = 0.5 \cdot 3^x$

27. $y = 2.5 \cdot 3^x$

28. $y = -2 \cdot 3^x$

29. $y = -4 \cdot 3^x$

30. $y = -\frac{1}{4} \cdot 3^x$

31. $y = -\frac{2}{3} \cdot 3^x$

32. $y = -0.5 \cdot 3^x$

33. $y = -2.5 \cdot 3^x$

34. ◆ MULTIPLE REPRESENTATIONS Given the function $y = a^x$, you can find x when $y = k$ by solving the *exponential equation* $k = a^x$. Use the following methods to solve $32 = 2^x$.

   a. Making a Table Make a table for the function using $x = 0, 1, 2, \ldots, 6$.

   b. Graphing Functions Graph the functions $y = 32$ and $y = 2^x$ on the same coordinate plane. Identify the x-coordinate of the intersection point.

   c. Using Powers Write 32 as a power of 2. Then use the fact that powers with the same base are equal provided that their exponents are equal.

35. ★ WRITING If a population triples each year, what is the population's growth rate (as a percent)? *Explain*.

36. CHALLENGE Write a linear function and an exponential function whose graphs pass through the points (0, 2) and (1, 6).

37. CHALLENGE *Compare* the graphs of the functions $f(x) = 2^{x+2}$ and $g(x) = 4 \cdot 2^x$. Use properties of exponents to explain your observations.

○ = See **WORKED-OUT SOLUTIONS** in Student Resources

★ = **STANDARDIZED TEST PRACTICE**

◆ = **MULTIPLE REPRESENTATIONS**

EXAMPLES 4 and 5
for Exs. 38–41

38. INVESTMENTS You deposit $125 in a savings account that earns 5% annual interest compounded yearly. Find the balance in the account after the given amounts of time.

a. 1 year   **b.** 2 years   **c.** 5 years   **d.** 20 years

39. MULTI-STEP PROBLEM One computer industry expert reported that there were about 600 million computers in use worldwide in 2001 and that the number was increasing at an annual rate of about 10%.

a. Write a function that models the number of computers in use over time.

b. Use the function to predict the number of computers that will be in use worldwide in 2009.

40. MULTI-STEP PROBLEM A research association reported that 3,173,000 gas grills were shipped by various manufacturers in the U.S. in 1985. Shipments increased by about 7% per year from 1985 to 2002.

a. Write a function that models the number of gas grills shipped over time.

b. About how many gas grills were shipped in 2002?

41. ◆◆ MULTIPLE REPRESENTATIONS A tree's cross-sectional area taken at a height of 4.5 feet from the ground is called its basal area and is measured in square inches. Tree growth can be measured by the growth of the tree's basal area. The initial basal area and annual growth rate for two particular trees are shown.


Tree 1
Growth rate: 6%
Initial basal area: 154 in.²

Tree 2
Growth rate: 10%
Initial basal area: 113 in.²
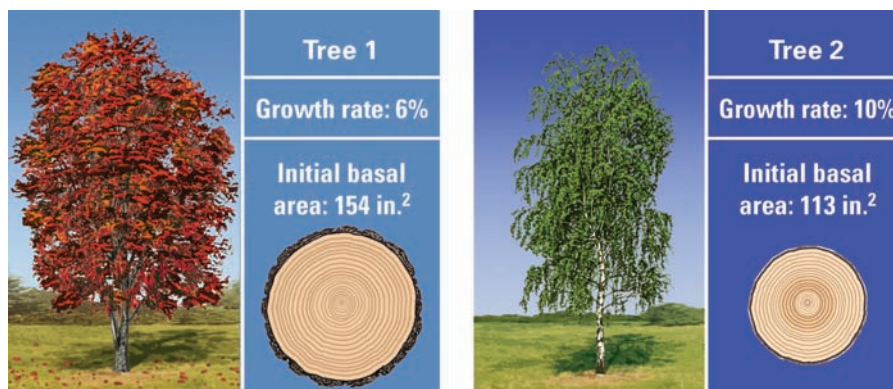
a. Writing a Model Write a function that models the basal area A of each tree over time.

b. Graphing a Function Use a graphing calculator to graph the functions from part (a) in the same coordinate plane. In about how many years will the trees have the same basal area?

42. ★ **SHORT RESPONSE** A company sells advertising blimps. The table shows the costs of advertising blimps of different lengths. Does the table represent an exponential function? *Explain*.

Length, ℓ (feet)	10	15	20	25
Cost, c (dollars)	400.00	700.00	1225.00	2143.75

43. ★ **MULTIPLE CHOICE** A weblog, or blog, refers to a website that contains a personal journal. According to one analyst, over one 18 month period, the number of blogs in existence doubled about every 6 months. The analyst estimated that there were about 600,000 blogs at the beginning of the period. How many blogs were there at the end of the period?

A 660,000   **B** 1,200,000   **C** 4,800,000   **D** 16,200,000

44. **TELECOMMUNICATIONS** For the period 1991–2001, the number y (in millions) of Internet users worldwide can be modeled by the function $y = 4.67(1.65)^x$ where x is the number of years since 1991.

a. Identify the initial amount, the growth factor, and the growth rate.

b. Graph the function. Identify its domain and range.

c. Use your graph from part (b) to graph the line $y = 21$. Estimate the year in which the number of Internet users worldwide was about 21 million.

45. **GRAPHING CALCULATOR** The frequency (in hertz) of a note played on a piano is a function of the position of the key that creates the note. The position of some piano keys and the frequencies of the notes created by the keys are shown below. Use the exponential regression feature on a graphing calculator to find an exponential model for the frequency of piano notes. What is the frequency of the note created by the 30th key?
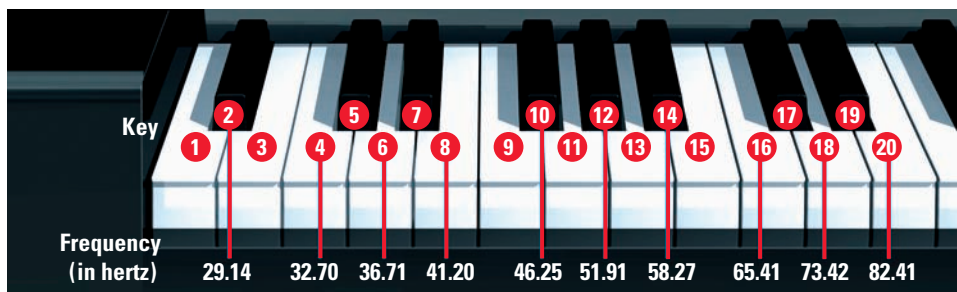
46. ★ **EXTENDED RESPONSE** In 1830, the population of the United States was 12,866,020. By 1890, the population was 62,947,714.

a. **Model** Assume the population growth from 1830 to 1890 was linear. Write a linear model for the U.S. population from 1830 to 1890. By about how much did the population grow per year from 1830 to 1890?

b. **Model** Assume the population growth from 1830 to 1890 was exponential. Write an exponential model for the U.S. population from 1830 to 1890. By approximately what percent did the population grow per year from 1830 to 1890?

c. **Explain** The U.S. population was 23,191,876 in 1850 and 38,558,371 in 1870. Which of the models in parts (a) and (b) is a better approximation of actual U.S. population for the time period 1850–1890? *Explain*.

★ = STANDARDIZED
TEST PRACTICE

COMPOUND INTEREST In Exercises 47–49, use the example below to find the balance of the account compounded with the given frequency.

> **EXAMPLE** Use the general compound interest formula
>
> **FINANCE** You deposit $1000 in an account that pays 3% annual interest. Find the balance after 8 years if the interest is compounded monthly.
>
> **Solution**
>
> The general formula for compound interest is $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$. In this formula, P is the initial amount, called principal, in an account that pays interest at an annual rate r and that is compounded n times per year. The amount A (in dollars) is the amount in the account after t years.
>
> Here, the interest is compounded monthly. So, $n = 12$.
>
> $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$ **Write compound interest formula.**
>
> $\quad = 1000\left(1 + \dfrac{0.03}{12}\right)^{12(8)}$ **Substitute 1000 for P, 0.03 for r, 12 for n, and 8 for t.**
>
> $\quad = 1000(1.0025)^{96}$ **Simplify.**
>
> $\quad \approx 1270.868467$ **Use a calculator.**
>
> ▶ The account balance after 8 years will be about $1270.87.

47. Yearly   **48.** Quarterly   **49.** Daily ($n = 365$)

50. ★ **WRITING** Which compounding frequency yields the highest balance in the account in the example above: monthly, yearly, quarterly, or daily? *Explain* why this is so.

51. **CHALLENGE** The value y (in dollars) of an investment of $1000 is given by $y = 1000(1.05)^t$ where t is the time in years. The *doubling time* is the value of t for which the amount invested doubles, so that $1000(1.05)^t = 2000$, or $(1.05)^t = 2$. Graph the functions $y = (1.05)^t$ and $y = 2$ on a graphing calculator. Estimate the doubling time.

Using ALTERNATIVE METHODS

Another Way to Solve Example 4

MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them.

MULTIPLE REPRESENTATIONS In Example 4, you saw how to solve a problem about the value of a collector car over time by using an exponential model. You can also solve the problem by using a spreadsheet.

PROBLEM

COLLECTOR CAR The owner of a 1953 Hudson Hornet convertible sold the car at an auction. The owner bought it in 1984 when its value was $11,000. The value of the car increased at a rate of 6.9% per year.

a. Write a function that models the value of the car over time.

b. The auction took place in 2004. What was the approximate value of the car at the time of the auction? Round your answer to the nearest dollar.

METHOD

Using a Spreadsheet An alternative approach is to use a spreadsheet.

a. The model for the value of the car over time is $C = 11{,}000(1.069)^t$, as shown in Example 4.

b. You can find the value of the car in 2004 by creating a spreadsheet.

STEP 1 **Create** a table showing the years since 1984 and the value of the car. Enter the car's value in 1984. To find the value in any year after 1984, multiply the car's value in the preceding year by the growth factor, as shown in cell B3 below.

FORMAT A SPREADSHEET
Format the spreadsheet so that calculations are rounded to 2 decimal places.

	A	B
1	Years since 1984, t	Value, C (dollars)
2	0	11000
3	1	=B2*1.069

STEP 2 **Find** the value of the car in 2004 by using the *fill down* feature until you get to the desired cell.

	A	B
1	Years since 1984, t	Value, C (dollars)
2	0	11000
3	1	11759
...	...	...
21	19	39081.31
22	20	41777.92

▶ From the spreadsheet, you can see the value of the car was about $41,778 in 2004.

WHAT IF? Suppose the owner decided to sell the car when it was worth about $28,000. In what year did the owner sell the car?

METHOD

Using a Spreadsheet To solve the equation algebraically, you need to substitute 28,000 for C and solve for t, but you have not yet learned how to solve this type of equation. An alternative to the algebraic approach is using a spreadsheet.

STEP 1 **Use** the same spreadsheet as on the previous page.

STEP 2 **Find** when the value of the car is about $28,000.

	A	B
	Years since 1984, t	Value, C (dollars)
1	Years since 1984, t	Value, C (dollars)
2	0	11000
...	...	...
15	13	26188.03
16	14	27995.01

> The value of the car is about $28,000 when $t = 14$.

▶ The owner sold the car in 1998.

PRACTICE

1. **TRANSPORTATION** In 1997 the average intercity bus fare for a particular state was $20. For the period 1997–2000, the bus fare increased at a rate of about 12% each year.

   a. Write a function that models the intercity bus fare for the period 1997–2000.

   b. Find the intercity bus fare in 1998. Use two different methods to solve the problem.

   c. In what year was the intercity bus fare $28.10? *Explain* how you found your answer.

2. **ERROR ANALYSIS** *Describe* and correct the error in writing the function for part (a) of Exercise 1.

   > Let b be the bus fare (in dollars) and t be the number of years since 1997.
   > $b = 20(0.12)^t$ ✗

3. **TECHNOLOGY** A computer's Central Processing Unit (CPU) is made up of transistors. One manufacturer released a CPU in May 1997 that had 7.5 million transistors. The number of transistors in the CPUs sold by the company increased at a rate of 3.9% per month.

   a. Write a function that models the number T (in millions) of transistors in the company's CPUs t months after May 1997.

   b. Use a spreadsheet to find the number of transistors in a CPU released by the company in November 2000.

4. **HOUSING** The value of a home in 2002 was $150,000. The value of the home increased at a rate of about 6.5% per year.

   a. Write a function that models the value of the home over time.

   b. Use a spreadsheet to find the year in which the value of the home was about $200,000.

# Exponential Models

MATERIALS • yarn • scissors

MATHEMATICAL PRACTICES

Reason abstractly and quantitatively.

QUESTION How can you model a situation using an exponential function?

EXPLORE Collect data so that you can write exponential models

STEP 1 *Fold and cut* Take about 1 yard of yarn and consider it to be 1 unit long. Fold it in half and cut, as shown. You are left with two pieces of yarn, each half the length of the original piece of yarn.
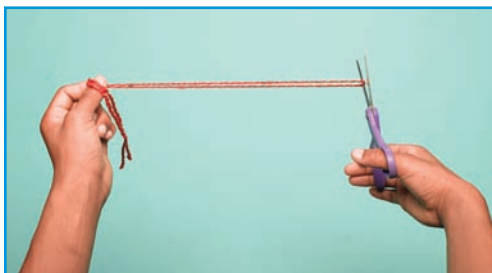
STEP 2 *Copy and complete* Copy the table. Notice that the row for stage 1 has the data from Step 1. For each successive stage, fold *all* the pieces of yarn in half and cut. Then record the number of new pieces and the length of each new piece until the table is complete.

Stage	Number of pieces	Length of each new piece
1	2	$\frac{1}{2}$
2	?	?
3	?	?
4	?	?
5	?	?

DRAW CONCLUSIONS Use your observations to complete these exercises

1. Use the data in the first and second columns of the table.
   a. Do the data represent an exponential function? *Explain* how you know.
   b. Write a function that models the number of pieces of yarn at stage *x*.
   c. Use the function to find the number of pieces of yarn at stage 10.

2. Use the data in the first and third columns of the table.
   a. Do the data represent an exponential function? *Explain* how you know.
   b. Write a function that models the length of each new piece of yarn at stage *x*.
   c. Use the function to find the length of each new piece of yarn at stage 10.

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