7.5 Write and Graph Exponential Decay Functions

Before	You wrote and graphed exponential growth functions.
Now	You will write and graph exponential decay functions.
Why?	So you can use a graph to solve a sports problem, as in Ex. 50.

Key Vocabulary
- **exponential decay**

A table of values represents an exponential function $y = ab^x$ provided successive y-values are multiplied by b each time the x-values increase by 1.

COMMON CORE

CC.9-12.A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.*

EXAMPLE 1 Write a function rule

Tell whether the table represents an exponential function. If so, write a rule for the function.

a.

+ 1 + 1 + 1

x	-1	0	1	2
y	$\frac{1}{9}$	$\frac{1}{3}$	1	3

$\times 3$ $\times 3$ $\times 3$

The y-values are multiplied by 3 for each increase of 1 in x, so the table represents an exponential function of the form $y = ab^x$ with $b = 3$.

The value of y when $x = 0$ is $\frac{1}{3}$, so $a = \frac{1}{3}$.

The table represents the exponential function $y = \frac{1}{3} \cdot 3^x$.

b.

+ 1 + 1 + 1

x	-1	0	1	2
y	4	1	$\frac{1}{4}$	$\frac{1}{16}$

$\times \frac{1}{4}$ $\times \frac{1}{4}$ $\times \frac{1}{4}$

The y-values are multiplied by $\frac{1}{4}$ for each increase of 1 in x, so the table represents an exponential function of the form $y = ab^x$ with $b = \frac{1}{4}$.

The value of y when $x = 0$ is 1, so $a = 1$.

The table represents the exponential function $y = \left(\frac{1}{4}\right)^x$.

✓ GUIDED PRACTICE for Example 1

1. Tell whether the table represents an exponential function. If so, write a rule for the function.

x	-1	0	1	2
y	5	1	$\frac{1}{5}$	$\frac{1}{25}$

EXAMPLE 2 Graph an exponential function

Graph the function $y = \left(\frac{1}{2}\right)^x$ and identify its domain and range.

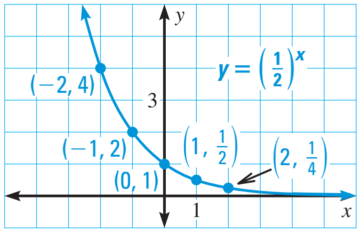
Solution

READ A GRAPH

Notice that the graph has a y-intercept of 1 and that it gets closer to the positive x-axis as the x-values increase.

STEP 1 **Make** a table of values. The domain is all real numbers.

x	-2	-1	0	1	2
y	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



STEP 2 **Plot** the points.

STEP 3 **Draw** a smooth curve through the points. From either the table or the graph, you can see the range is all positive real numbers.

EXAMPLE 3 Compare graphs of exponential functions

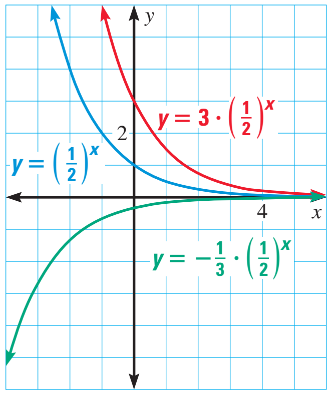
Graph the functions $y = 3 \cdot \left(\frac{1}{2}\right)^x$ and $y = -\frac{1}{3} \cdot \left(\frac{1}{2}\right)^x$. Compare each graph with the graph of $y = \left(\frac{1}{2}\right)^x$.

DESCRIBE A FUNCTION

An exponential decay function has an unbroken graph, so the function is continuous.

Solution

x	$y = \left(\frac{1}{2}\right)^x$	$y = 3 \cdot \left(\frac{1}{2}\right)^x$	$y = -\frac{1}{3} \cdot \left(\frac{1}{2}\right)^x$
-2	4	12	$-\frac{4}{3}$
-1	2	6	$-\frac{2}{3}$
0	1	3	$-\frac{1}{3}$
1	$\frac{1}{2}$	$\frac{3}{2}$	$-\frac{1}{6}$
2	$\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{12}$



Because the y-values for $y = 3 \cdot \left(\frac{1}{2}\right)^x$ are 3 times the corresponding y-values for $y = \left(\frac{1}{2}\right)^x$, the graph of $y = 3 \cdot \left(\frac{1}{2}\right)^x$ is a vertical stretch of the graph of $y = \left(\frac{1}{2}\right)^x$.

Because the y-values for $y = -\frac{1}{3} \cdot \left(\frac{1}{2}\right)^x$ are $-\frac{1}{3}$ times the corresponding y-values for $y = \left(\frac{1}{2}\right)^x$, the graph of $y = -\frac{1}{3} \cdot \left(\frac{1}{2}\right)^x$ is a vertical shrink with reflection in the x-axis of the graph of $y = \left(\frac{1}{2}\right)^x$.

GUIDED PRACTICE for Examples 2 and 3

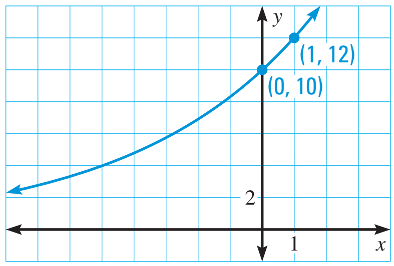
2. Graph $y = (0.4)^x$ and identify its domain and range.
3. Graph $y = 5 \cdot (0.4)^x$. Compare the graph with the graph of $y = (0.4)^x$.

COMPARE GRAPHS When $a > 0$ and $0 < b < 1$, the function $y = ab^x$ represents **exponential decay**. The graph of an exponential decay function falls from left to right. In comparison, the graph of an exponential growth function $y = ab^x$ where $a > 0$ and $b > 1$ rises from the left.

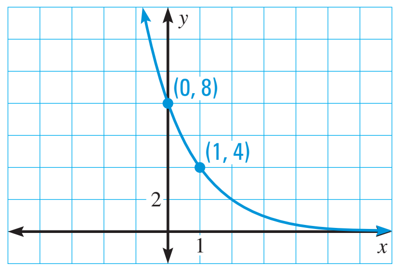
EXAMPLE 4 Classify and write rules for functions

Tell whether the graph represents *exponential growth* or *exponential decay*. Then write a rule for the function.

a.



b.



Solution

a. The graph represents exponential growth ($y = ab^x$ where $b > 1$). The y-intercept is 10, so $a = 10$. Find the value of b by using the point (1, 12) and $a = 10$.

$y = ab^x$	**Write function.**
$12 = 10 \cdot b^1$	**Substitute.**
$1.2 = b$	**Solve.**

A function rule is $y = 10(1.2)^x$.

b. The graph represents exponential decay ($y = ab^x$ where $0 < b < 1$). The y-intercept is 8, so $a = 8$. Find the value of b by using the point (1, 4) and $a = 8$.

$y = ab^x$	**Write function.**
$4 = 8 \cdot b^1$	**Substitute.**
$0.5 = b$	**Solve.**

A function rule is $y = 8(0.5)^x$.

ANALYZE GRAPHS
For the function $y = ab^x$, where $x = 0$, the value of y is $y = ax^0 = a$. This means that the graph of $y = ab^x$ has a y-intercept of a.

✓ GUIDED PRACTICE for Example 4

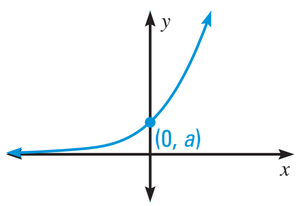
4. The graph of an exponential function passes through the points (0, 10) and (1, 8). Graph the function. Tell whether the graph represents *exponential growth* or *exponential decay*. Write a rule for the function.

CONCEPT SUMMARY *For Your Notebook*

Exponential Growth and Decay

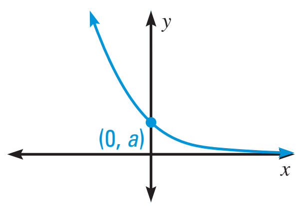
Exponential Growth

$y = ab^x$, $a > 0$ and $b > 1$



Exponential Decay

$y = ab^x$, $a > 0$ and $0 < b < 1$



EXPONENTIAL DECAY When a quantity decays exponentially, it decreases by the same percent over equal time periods. To find the amount of the quantity left after t time periods, use the following model.

KEY CONCEPT *For Your Notebook*

Exponential Decay Model

a is the **initial amount**. r is the **decay rate**.

$$y = a(\underbrace{1 - r}_{})^t$$

$1 - r$ is the **decay factor**. t is the **time period**.

REWRITE EQUATIONS
Notice that you can rewrite $y = ab^x$ as $y = a(1 - r)^t$ by replacing b with $1 - r$ and x with t (for time).

The relationship between the decay rate r and the decay factor $1 - r$ is similar to the relationship between the growth rate and growth factor in an exponential growth model. You will explore this relationship in Exercise 45.

EXAMPLE 5 Solve a multi-step problem

FORESTRY The number of acres of Ponderosa pine forests decreased in the western United States from 1963 to 2002 by 0.5% annually. In 1963 there were about 41 million acres of Ponderosa pine forests.

a. Write a function that models the number of acres of Ponderosa pine forests in the western United States over time.

b. To the nearest tenth, about how many million acres of Ponderosa pine forests were there in 2002?

Solution

a. Let P be the number of acres (in millions), and let t be the time (in years) since 1963. The initial value is 41, and the decay rate is 0.005.

$P = a(1 - r)^t$ **Write exponential decay model.**

$= 41(1 - 0.005)^t$ **Substitute 41 for *a* and 0.005 for *r*.**

$= 41(0.995)^t$ **Simplify.**

AVOID ERRORS
The decay rate in this example is 0.5%, or 0.005. So, the decay factor is $1 - 0.005$, or 0.995, not 0.005.

b. To find the number of acres in 2002, 39 years after 1963, substitute 39 for t.

$P = 41(0.995)^{39} \approx 33.7$ **Substitute 39 for *t*. Use a calculator.**

▶ There were about 33.7 million acres of Ponderosa pine forests in 2002.

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✓ **GUIDED PRACTICE** for Example 5

5. WHAT IF? In Example 5, suppose the decay rate of the forests remains the same beyond 2002. About how many acres will be left in 2010?

7.5 EXERCISES

HOMEWORK KEY

○ = See **WORKED-OUT SOLUTIONS** Exs. 7 and 49

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 19, 36, 45, and 49

◆ = **MULTIPLE REPRESENTATIONS** Ex. 50

SKILL PRACTICE

1. **VOCABULARY** What is the decay factor in the exponential decay model $y = a(1 - r)^t$?

2. ★ **WRITING** *Explain* how you can tell if a graph represents *exponential growth* or *exponential decay.*

EXAMPLE 1 for Exs. 3–6

WRITING FUNCTIONS **Tell whether the table represents an exponential function. If so, write a rule for the function.**

3.

x	–1	0	1	2
y	2	8	32	128

4.

x	–1	0	1	2
y	50	10	2	0.4

5.

x	–1	0	1	2
y	6	2	$\frac{2}{3}$	$\frac{2}{9}$

6.

x	–1	0	1	2
y	–11	–7	–3	1

EXAMPLE 2 for Exs. 7–18

GRAPHING FUNCTIONS **Graph the function and identify its domain and range.**

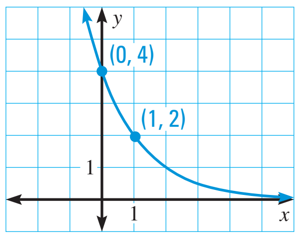
7. $y = \left(\frac{1}{5}\right)^x$
8. $y = \left(\frac{1}{6}\right)^x$
9. $y = \left(\frac{2}{3}\right)^x$
10. $y = \left(\frac{3}{4}\right)^x$
11. $y = \left(\frac{4}{5}\right)^x$
12. $y = \left(\frac{3}{5}\right)^x$
13. $y = (0.3)^x$
14. $y = (0.5)^x$
15. $y = (0.1)^x$
16. $y = (0.9)^x$
17. $y = (0.7)^x$
18. $y = (0.25)^x$

EXAMPLE 3 for Exs. 19–31

19. ★ **MULTIPLE CHOICE** The graph of which function is shown?

(A) $y = (0.25)^x$ (B) $y = (0.5)^x$

(C) $y = 0.25 \cdot (0.5)^x$ (D) $y = 4 \cdot (0.5)^x$



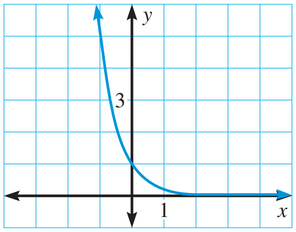
COMPARING FUNCTIONS **Graph the function. Compare the graph with the graph of $y = \left(\frac{1}{4}\right)^x$.**

20. $y = 5 \cdot \left(\frac{1}{4}\right)^x$
21. $y = 3 \cdot \left(\frac{1}{4}\right)^x$
22. $y = \frac{1}{2} \cdot \left(\frac{1}{4}\right)^x$
23. $y = \frac{1}{3} \cdot \left(\frac{1}{4}\right)^x$
24. $y = 0.2 \cdot \left(\frac{1}{4}\right)^x$
25. $y = 1.5 \cdot \left(\frac{1}{4}\right)^x$
26. $y = -5 \cdot \left(\frac{1}{4}\right)^x$
27. $y = -3 \cdot \left(\frac{1}{4}\right)^x$
28. $y = -\frac{1}{2} \cdot \left(\frac{1}{4}\right)^x$
29. $y = -\frac{1}{3} \cdot \left(\frac{1}{4}\right)^x$
30. $y = -0.2 \cdot \left(\frac{1}{4}\right)^x$
31. $y = -1.5 \cdot \left(\frac{1}{4}\right)^x$

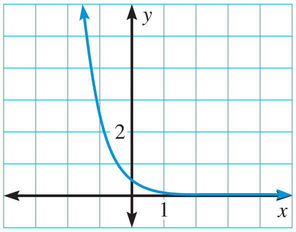
MATCHING **Match the function with its graph.**

32. $y = (0.2)^x$ **33.** $y = 5(0.2)^x$ **34.** $y = \frac{1}{2}(0.2)^x$

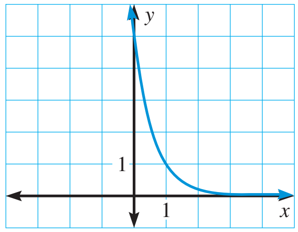
A.



B.



C.



35. **POPULATION** A population of 90,000 decreases by 2.5% per year. Identify the initial amount, the decay factor, and the decay rate. Then write a function that models the population over time.

36. ★ **MULTIPLE CHOICE** What is the decay rate of the function $y = 4(0.97)^t$?

Ⓐ 4 Ⓑ 0.97 Ⓒ 0.3 Ⓓ 0.03

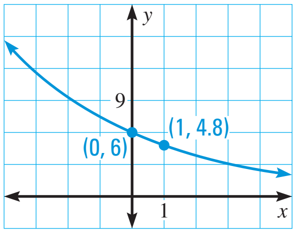
37. **ERROR ANALYSIS** In 2004 a person purchased a car for $25,000. The value of the car decreased by 14% annually. *Describe* and correct the error in writing a function that models the value of the car since 2004.

$$y = a(1 - r)^t$$
$$= 25{,}000(0.14)^t$$

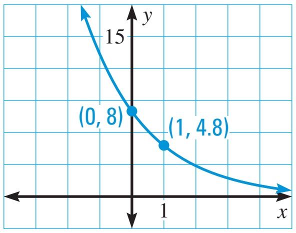
EXAMPLE 4
for Exs. 38–40

RECOGNIZING EXPONENTIAL MODELS **Tell whether the graph represents *exponential growth* or *exponential decay*. Then write a rule for the function.**

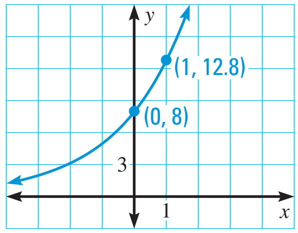
38.



39.



40.



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41. **REASONING** Without graphing, explain how the graphs of the given functions are related to the graph of $f(x) = (0.5)^x$.

a. $m(x) = \frac{1}{3} \cdot (0.5)^x$ **b.** $n(x) = -4 \cdot (0.5)^x$ **c.** $p(x) = (0.5)^x + 1$

CHALLENGE **Write an exponential function of the form $y = ab^x$ whose graph passes through the given points.**

42. $(0, 1), \left(2, \frac{1}{4}\right)$ **43.** $(1, 20), (2, 4)$ **44.** $\left(1, \frac{3}{2}\right), \left(2, \frac{3}{4}\right)$

45. ★ **WRITING** The *half-life* of a radioactive substance is the time required for half the substance to decay. The amount A (in grams) of a 100 gram sample of a radioactive substance remaining after t half-lives is given by $A = 100(0.5)^t$. Suppose the substance has a half-life of 10 days. *Explain* how to find the amount left after 40 days. Then find the amount.

46. **CHALLENGE** *Compare* the graphs of the functions $f(x) = 4^{x-2}$ and $g(x) = \frac{1}{16} \cdot 4^x$. Use properties of exponents to explain your observation.

○ = See **WORKED-OUT SOLUTIONS** in Student Resources ★ = **STANDARDIZED TEST PRACTICE** = **MULTIPLE REPRESENTATIONS**

PROBLEM SOLVING

GRAPHING CALCULATOR **You may wish to use a graphing calculator to complete the following Problem Solving exercises.**

EXAMPLE 5 for Exs. 47–50

47. CELL PHONES You purchase a cell phone for $125. The value of the cell phone decreases by about 20% annually. Write a function that models the value of the cell phone over time. Then find the value of the cell phone after 3 years.

48. ANIMAL POPULATION Scientists studied the population of a species of bat in some caves in Missouri from 1983 to 2003. In 1983, there were 141,200 bats living in the caves. That number decreased by about 11% annually until 2003.

a. Identify the initial amount, the decay factor, and the decay rate.

b. Write a function that models the number of bats since 1983. Then find the number of bats in 2003.

49. ★ **SHORT RESPONSE** In 2003 a family bought a boat for $4000. The boat depreciates (loses value) at a rate of 7% annually. In 2006 a person offers to buy the boat for $3000. Should the family sell the boat? *Explain*.

50. **MULTIPLE REPRESENTATIONS** There are a total of 128 teams at the start of a citywide 3-on-3 basketball tournament. Half of the teams are eliminated after each round.

a. **Writing a Model** Write a function for the number of teams left after x rounds.

b. **Making a Table** Make a table for the function using $x = 0, 1, 2, \ldots, 7$.

c. **Drawing a Graph** Use the table in part (b) to graph the function. After which round are there 4 teams left in the tournament?

51. GUITARS The frets on a guitar are the small metal bars that divide the fingerboard. The distance d (in inches) between the nut and the first fret or any two consecutive frets can be modeled by the function $d = 1.516(0.9439)^f$ where f is the number of the fret farthest from the nut.

a. Identify the decay factor and the decay rate for the model.

b. What is the distance between the nut and the first fret?

c. The distance between the 12th and 13th frets is about half the distance between the nut and the first fret. Use this fact to find the distance between the 12th and 13th frets. Use the model to verify your answer.

52. **CHALLENGE** A college student finances a computer that costs \$1850. The financing plan states that as long as a minimum monthly payment of 2.25% of the remaining balance is made, the student does not have to pay interest for 24 months. The student makes only the minimum monthly payments until the last payment. What is the amount of the last payment if the student buys the computer without paying interest? Round your answer to the nearest cent.

53. **MULTI-STEP PROBLEM** Maximal oxygen consumption is the maximum volume of oxygen (in liters per minute) that the body uses during exercise. Maximal oxygen consumption varies from person to person and decreases with age by about 0.5% per year after age 25 for active adults.

a. Model A 25-year-old female athlete has a maximal oxygen consumption of 4 liters per minute. Another 25-year-old female athlete has a maximal oxygen consumption of 3.5 liters per minute. Write a function for each athlete that models the maximal consumption each year after age 25.

b. Graph Graph the models in the same coordinate plane.

c. Estimate About how old will the first athlete be when her maximal oxygen consumption is equal to what the second athlete's maximal oxygen consumption is at age 25?

QUIZ

Simplify the expression. Write your answer using only positive exponents.

1. $(-4x)^4 \cdot (-4)^{-6}$ **2.** $(-3x^7y^{-2})^{-3}$ **3.** $\frac{1}{(5z)^{-3}}$ **4.** $\frac{(6x)^{-2}y^5}{-x^3y^{-7}}$

Graph the function.

5. $y = \left(\frac{5}{2}\right)^x$ **6.** $y = 3 \cdot \left(\frac{1}{4}\right)^x$ **7.** $y = \frac{1}{4} \cdot 3^x$

8. $y = (0.1)^x$ **9.** $y = 10 \cdot 5^x$ **10.** $y = 7(0.4)^x$

11. **COINS** You purchase a coin from a coin collector for \$25. Each year the value of the coin increases by 8%. Write a function that models the value of the coin over time. Then find the value of the coin after 10 years. Round to the nearest cent.

See **EXTRA PRACTICE** in Student Resources **ONLINE QUIZ** at my.hrw.com