8.8 Factor Polynomials Completely

Before You factored polynomials.

Now You will factor polynomials completely.

Why? So you can model the height of a projectile, as in Ex. 71.

Key Vocabulary
- **factor by grouping**
- **factor completely**

You have used the distributive property to factor a greatest common monomial from a polynomial. Sometimes, you can factor out a common binomial.

EXAMPLE 1 Factor out a common binomial

COMMON CORE

CC.9-12.A.SSE.3a Factor a quadratic expression to reveal the zeros of the function it defines.

Factor the expression.

a. $2x(x + 4) - 3(x + 4)$ **b.** $3y^2(y - 2) + 5(2 - y)$

Solution

a. $2x(x + 4) - 3(x + 4) = (x + 4)(2x - 3)$

b. The binomials $y - 2$ and $2 - y$ are opposites. Factor -1 from $2 - y$ to obtain a common binomial factor.

$3y^2(y - 2) + 5(2 - y) = 3y^2(y - 2) - 5(y - 2)$ Factor -1 from $(2 - y)$.

$= (y - 2)(3y^2 - 5)$ Distributive property

GROUPING You may be able to use the distributive property to factor polynomials with four terms. Factor a common monomial from pairs of terms, then look for a common binomial factor. This is called **factor by grouping**.

EXAMPLE 2 Factor by grouping

Factor the polynomial.

a. $x^3 + 3x^2 + 5x + 15$ **b.** $y^2 + y + yx + x$

Solution

a. $x^3 + 3x^2 + 5x + 15 = (x^3 + 3x^2) + (5x + 15)$ Group terms.

$= x^2(x + 3) + 5(x + 3)$ Factor each group.

$= (x + 3)(x^2 + 5)$ Distributive property

b. $y^2 + y + yx + x = (y^2 + y) + (yx + x)$ Group terms.

$= y(y + 1) + x(y + 1)$ Factor each group.

$= (y + 1)(y + x)$ Distributive property

CHECK WORK
Remember that you can check a factorization by multiplying the factors.

EXAMPLE 3 Factor by grouping

Factor $x^3 - 6 + 2x - 3x^2$.

Solution

The terms x^3 and -6 have no common factor. Use the commutative property to rearrange the terms so that you can group terms with a common factor.

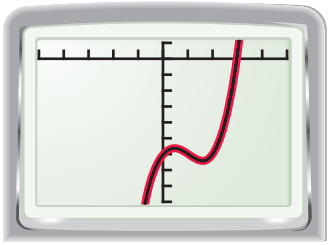
$$x^3 - 6 + 2x - 3x^2 = x^3 - 3x^2 + 2x - 6 \quad \text{Rearrange terms.}$$

$$= (x^3 - 3x^2) + (2x - 6) \quad \text{Group terms.}$$

$$= x^2(x - 3) + 2(x - 3) \quad \text{Factor each group.}$$

$$= (x - 3)(x^2 + 2) \quad \text{Distributive property}$$

CHECK Check your factorization using a graphing calculator. Graph $y_1 = x^3 - 6 + 2x - 3x^2$ and $y_2 = (x - 3)(x^2 + 2)$. Because the graphs coincide, you know that your factorization is correct.



GUIDED PRACTICE for Examples 1, 2, and 3

Factor the expression.

1. $x(x - 2) + (x - 2)$ **2.** $a^3 + 3a^2 + a + 3$ **3.** $y^2 + 2x + yx + 2y$

READING
If a polynomial has two or more terms and is unfactorable, it is called a *prime polynomial*.

FACTORING COMPLETELY You have seen that the polynomial $x^2 - 1$ can be factored as $(x + 1)(x - 1)$. This polynomial is factorable. Notice that the polynomial $x^2 + 1$ cannot be written as the product of polynomials with integer coefficients. This polynomial is unfactorable. A factorable polynomial with integer coefficients is **factored completely** if it is written as a product of unfactorable polynomials with integer coefficients.

CONCEPT SUMMARY *For Your Notebook*

Guidelines for Factoring Polynomials Completely

To factor a polynomial completely, you should try each of these steps.

1. Factor out the greatest common monomial factor.	$3x^2 + 6x = 3x(x + 2)$
2. Look for a difference of two squares or a perfect square trinomial.	$x^2 + 4x + 4 = (x + 2)^2$
3. Factor a trinomial of the form $ax^2 + bx + c$ into a product of binomial factors.	$3x^2 - 5x - 2 = (3x + 1)(x - 2)$
4. Factor a polynomial with four terms by grouping.	$x^3 + x - 4x^2 - 4 = (x^2 + 1)(x - 4)$

EXAMPLE 4 Factor completely

Factor the polynomial completely.

a. $n^2 + 2n - 1$ **b.** $4x^3 - 44x^2 + 96x$ **c.** $50h^4 - 2h^2$

Solution

a. The terms of the polynomial have no common monomial factor. Also, there are no factors of -1 that have a sum of 2. This polynomial cannot be factored.

b. $4x^3 - 44x^2 + 96x = 4x(x^2 - 11x + 24)$ **Factor out $4x$.**

$= 4x(x - 3)(x - 8)$ **Find two negative factors of 24 that have a sum of -11.**

c. $50h^4 - 2h^2 = 2h^2(25h^2 - 1)$ **Factor out $2h^2$.**

$= 2h^2(5h - 1)(5h + 1)$ **Difference of two squares pattern**

✓ GUIDED PRACTICE for Example 4

Factor the polynomial completely.

4. $3x^3 - 12x$ **5.** $2y^3 - 12y^2 + 18y$ **6.** $m^3 - 2m^2 - 8m$

EXAMPLE 5 Solve a polynomial equation

Solve $3x^3 + 18x^2 = -24x$.

$3x^3 + 18x^2 = -24x$ **Write original equation.**

$3x^3 + 18x^2 + 24x = 0$ **Add $24x$ to each side.**

$3x(x^2 + 6x + 8) = 0$ **Factor out $3x$.**

$3x(x + 2)(x + 4) = 0$ **Factor trinomial.**

$3x = 0$ *or* $x + 2 = 0$ *or* $x + 4 = 0$ **Zero-product property**

$x = 0$ $x = -2$ $x = -4$ **Solve for x.**

▸ The solutions of the equation are 0, -2, and -4.

CHECK Check each solution by substituting it for x in the equation. One check is shown here.

$3(-2)^3 + 18(-2)^2 \stackrel{?}{=} -24(-2)$

$-24 + 72 \stackrel{?}{=} 48$

$48 = 48$ ✓

✓ GUIDED PRACTICE for Example 5

Solve the equation.

7. $w^3 - 8w^2 + 16w = 0$ **8.** $x^3 - 25x = 0$ **9.** $c^3 - 7c^2 + 12c = 0$

EXAMPLE 6 Solve a multi-step problem

TERRARIUM A terrarium in the shape of a rectangular prism has a volume of 4608 cubic inches. Its length is more than 10 inches. The dimensions of the terrarium are shown. Find the length, width, and height of the terrarium.

Solution

STEP 1 **Write** a verbal model. Then write an equation.

Volume (cubic inches)	=	Length (inches)	·	Width (inches)	·	Height (inches)
↓		↓		↓		↓
4608	$=$	$(36 - w)$	$\cdot$	w	$\cdot$	$(w + 4)$

STEP 2 **Solve** the equation for w.

$4608 = (36 - w)(w)(w + 4)$ **Write equation.**

$0 = 32w^2 + 144w - w^3 - 4608$ **Multiply. Subtract 4608 from each side.**

$0 = (-w^3 + 32w^2) + (144w - 4608)$ **Group terms.**

$0 = -w^2(w - 32) + 144(w - 32)$ **Factor each group.**

$0 = (w - 32)(-w^2 + 144)$ **Distributive property**

$0 = -1(w - 32)(w^2 - 144)$ **Factor -1 from $-w^2 + 144$.**

$0 = -1(w - 32)(w - 12)(w + 12)$ **Difference of two squares pattern**

$w - 32 = 0$ *or* $w - 12 = 0$ *or* $w + 12 = 0$ **Zero-product property**

$w = 32$ $w = 12$ $w = -12$ **Solve for w.**

STEP 3 **Choose** the solution of the equation that is the correct value of w. Disregard $w = -12$, because the width cannot be negative.

You know that the length is more than 10 inches. Test the solutions 12 and 32 in the expression for the length.

Length $= 36 - 12 = 24$ ✓ *or* Length $= 36 - 32 = 4$ ✗

The solution 12 gives a length of 24 inches, so 12 is the correct value of w.

STEP 4 **Find** the height.

Height $= w + 4 = 12 + 4 = 16$

▶ The width is 12 inches, the length is 24 inches, and the height is 16 inches.

GUIDED PRACTICE for Example 6

10. **DIMENSIONS OF A BOX** A box in the shape of a rectangular prism has a volume of 72 cubic feet. The box has a length of x feet, a width of $(x - 1)$ feet, and a height of $(x + 9)$ feet. Find the dimensions of the box.

8.8 EXERCISES

HOMEWORK KEY

○ = See **WORKED-OUT SOLUTIONS** Exs. 13, 23, and 71

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 12, 41, 55, 71, and 73

SKILL PRACTICE

1. **VOCABULARY** What does it mean for a polynomial to be factored completely?

2. ★ **WRITING** *Explain* how you know if a polynomial is unfactorable.

EXAMPLE 1 for Exs. 3–12

BINOMIAL FACTORS **Factor the expression.**

3. $x(x - 8) + (x - 8)$
4. $5y(y + 3) - 2(y + 3)$
5. $6z(z - 4) - 7(z - 4)$
6. $10(a - 6) - 3a(a - 6)$
7. $b^2(b + 5) - 3(b + 5)$
8. $7c^2(c + 9) + 2(c + 9)$
9. $x(13 + x) - (x + 13)$
10. $y^2(y - 4) + 5(4 - y)$
11. $12(z - 1) - 5z^2(1 - z)$

12. ★ **MULTIPLE CHOICE** Which is the correct factorization of $x^2(x - 8) + 5(8 - x)$?

Ⓐ $(x^2 + 5)(x - 8)$ Ⓑ $(x^2 + 5)(8 - x)$

Ⓒ $(x^2 - 5)(x - 8)$ Ⓓ $(x^2 - 5)(8 - x)$

EXAMPLES 2 and 3 for Exs. 13–22

FACTORING BY GROUPING **Factor the polynomial.**

13. $x^3 + x^2 + 2x + 2$
14. $y^3 - 9y^2 + y - 9$
15. $z^3 - 4z^2 + 3z - 12$
16. $c^3 + 7c^2 + 5c + 35$
17. $a^3 + 13a^2 - 5a - 65$
18. $2s^3 - 3s^2 + 18s - 27$
19. $5n^3 - 4n^2 + 25n - 20$
20. $x^2 + 8x - xy - 8y$
21. $y^2 + y + 5xy + 5x$

22. **ERROR ANALYSIS** *Describe* and correct the error in factoring.

$$a^3 + 8a^2 - 6a - 48 = a^2(a + 8) + 6(a + 8)$$
$$= (a + 8)(a^2 + 6)$$

EXAMPLE 4 for Exs. 23–42

FACTORING COMPLETELY **Factor the polynomial completely.**

23. $x^4 - x^2$
24. $36a^4 - 4a^2$
25. $3n^5 - 48n^3$
26. $4y^6 - 16y^4$
27. $75c^9 - 3c^7$
28. $72p - 2p^3$
29. $32s^4 - 8s^2$
30. $80z^8 - 45z^6$
31. $m^2 - 5m - 35$
32. $6g^3 - 24g^2 + 24g$
33. $3w^4 + 24w^3 + 48w^2$
34. $3r^5 + 3r^4 - 90r^3$
35. $b^3 - 5b^2 - 4b + 20$
36. $h^3 + 4h^2 - 25h - 100$
37. $9t^3 + 18t - t^2 - 2$
38. $2x^5y - 162x^3y$
39. $7a^3b^3 - 63ab^3$
40. $-4s^3t^3 + 24s^2t^2 - 36st$

41. ★ **MULTIPLE CHOICE** What is the completely factored form of $3x^6 - 75x^4$?

Ⓐ $3x^4(x^2 - 25)$ Ⓑ $3x^4(x - 5)^2$ Ⓒ $3x^4(x + 5)^2$ Ⓓ $3x^4(x - 5)(x + 5)$

42. **ERROR ANALYSIS** *Describe* and correct the error in factoring the polynomial completely.

$$x^3 - 6x^2 - 9x + 54 = x^2(x - 6) - 9(x - 6)$$
$$= (x - 6)(x^2 - 9)$$

EXAMPLE 5 for Exs. 43–54

SOLVING EQUATIONS **Solve the equation.**

43. $x^3 + x^2 - 4x - 4 = 0$ **44.** $a^3 - 11a^2 - 9a + 99 = 0$ **45.** $4y^3 - 7y^2 - 16y + 28 = 0$

46. $5n^3 - 30n^2 + 40n = 0$ **47.** $3b^3 + 24b^2 + 45b = 0$ **48.** $2t^5 + 2t^4 - 144t^3 = 0$

49. $z^3 - 81z = 0$ **50.** $c^4 - 100c^2 = 0$ **51.** $12s - 3s^3 = 0$

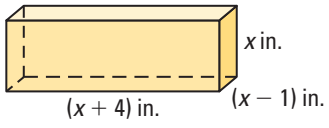
52. $2x^3 - 10x^2 + 40 = 8x$ **53.** $3p + 1 = p^2 + 3p^3$ **54.** $m^3 - 3m^2 = 4m - 12$

55. ★ **WRITING** Is it possible to find three solutions of the equation $x^3 + 2x^2 + 3x + 6 = 0$? *Explain* why or why not.

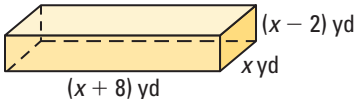
GEOMETRY **Find the length, width, and height of the rectangular prism with the given volume.**

HINT
In Ex. 57, convert the given volume to cubic yards. Use the conversion factor $\frac{1 \text{ yd}^3}{27 \text{ ft}^3}$.

56. Volume = 12 cubic inches



57. Volume: 2592 cubic feet



FACTORING COMPLETELY **Factor the polynomial completely.**

58. $x^3 + 2x^2y - x - 2y$ **59.** $8b^3 - 4b^2a - 18b + 9a$ **60.** $4s^2 - s + 12st - 3t$

FACTOR BY GROUPING **In Exercises 61–66, use the example below to factor the trinomial by grouping.**

EXAMPLE **Factor a trinomial by grouping**

Factor $8x^2 + 10x - 3$ by grouping.

Solution

Notice that the polynomial is in the form $ax^2 + bx + c$.

STEP 1 **Write** the product ac as the product of two factors that have a sum of b. In this case, the product ac is $8(-3) = -24$. Find two factors of -24 that have a sum of 10.

$-24 = 12 \cdot (-2)$ and $12 + (-2) = 10$

STEP 2 **Rewrite** the middle term as two terms with coefficients 12 and -2.

$8x^2 + \mathbf{10x} - 3 = 8x^2 \mathbf{+ 12x - 2x} - 3$

STEP 3 **Factor** by grouping.

$8x^2 + 12x - 2x - 3 = (8x^2 + 12x) + (-2x - 3)$ **Group terms.**

$= 4x(2x + 3) - (2x + 3)$ **Factor each group.**

$= (2x + 3)(4x - 1)$ **Distributive property**

61. $6x^2 + 5x - 4$ **62.** $10s^2 + 19s + 6$ **63.** $12n^2 - 13n + 3$

64. $16a^2 + 14a + 3$ **65.** $21w^2 + 8w - 4$ **66.** $15y^2 - 31y + 10$

67. **CHALLENGE** Use factoring by grouping to show that a trinomial of the form $a^2 + 2ab + b^2$ can be factored as $(a + b)^2$. *Justify* your steps.

PROBLEM SOLVING

EXAMPLE 6 for Exs. 68–70

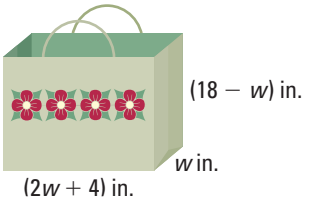
68. CYLINDRICAL VASE A vase in the shape of a cylinder has a height of 6 inches and a volume of 24π cubic inches. What is the radius of the vase?

69. CARPENTRY You are building a birdhouse that will have a volume of 128 cubic inches. The birdhouse will have the dimensions shown.

a. Write a polynomial that represents the volume of the birdhouse.

b. What are the dimensions of the birdhouse?

70. BAG SIZE A gift bag is shaped like a rectangular prism and has a volume of 1152 cubic inches. The dimensions of the gift bag are shown. The height is greater than the width. What are the dimensions of the gift bag?

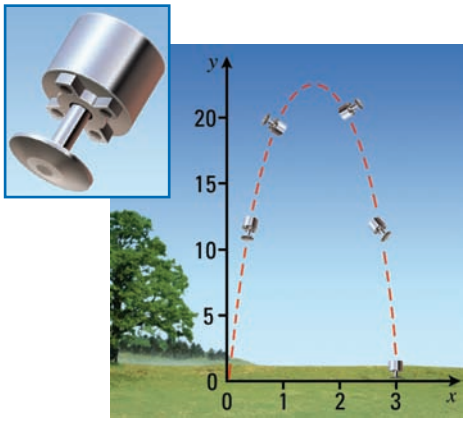


71. ★ SHORT RESPONSE A pallino is the small target ball that is tossed in the air at the beginning of a game of bocce. The height h (in meters) of the pallino after you throw it can be modeled by $h = -4.9t^2 + 3.9t + 1$ where t is the time (in seconds) since you released it.

a. Find the zeros of the function.

b. Do the zeros of the function have any meaning in this situation? *Explain* your reasoning.

72. JUMPING ROBOT The path of a jumping robot can be modeled by the graph of the equation $y = -10x^2 + 30x$ where x and y are both measured in feet. On a coordinate plane, the ground is represented by the x-axis, and the robot's starting position is the origin.



a. The robot's maximum height is 22.5 feet. What is the robot's horizontal distance from its starting point when its height is 22.5 feet?

b. How far has the robot traveled horizontally when it lands on the ground? *Explain* your answer.

73. ★ EXTENDED RESPONSE The width of a box is 4 inches more than the height h. The length is the difference of 9 inches and the height.

a. Write a polynomial that represents the volume of the box.

b. The volume of the box is 180 cubic inches. What are all the possible dimensions of the box?

c. Which dimensions result in a box with the smallest possible surface area? *Explain* your reasoning.

◯ = See WORKED-OUT SOLUTIONS in Student Resources

★ = STANDARDIZED TEST PRACTICE

74. CHALLENGE A plastic cube is used to display an autographed baseball. The cube has an outer surface area of 54 square inches.

a. What is the length of an outer edge of the cube?

b. What is the greatest volume the cube can possibly have? *Explain* why the actual volume inside of the cube may be less than the greatest possible volume.

QUIZ

Factor the polynomial.

1. $x^2 - 400$ **2.** $18 - 32z^2$ **3.** $169x^2 - 25y^2$

4. $n^2 - 6n + 9$ **5.** $100a^2 + 20a + 1$ **6.** $8r^2 - 40rs + 50s^2$

Factor the polynomial completely.

7. $3x^5 - 75x^3$ **8.** $72s^4 - 8s^2$ **9.** $3x^4y - 300x^2y$

10. $a^3 - 4a^2 - 21a$ **11.** $2h^4 + 28h^3 + 98h^2$ **12.** $z^3 - 4z^2 - 16z + 64$

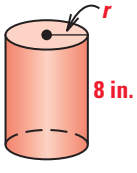
Solve the equation.

13. $x^2 + 10x + 25 = 0$ **14.** $48 - 27m^2 = 0$

15. $w^3 - w^2 - 4w + 4 = 0$ **16.** $4x^3 - 28x^2 + 40x = 0$

17. $3x^5 - 6x^4 - 45x^3 = 0$ **18.** $x^3 - 121x = 0$

19. VOLUME The cylinder shown has a volume of 72π cubic inches.



a. Write a polynomial that represents the volume of the cylinder. Leave your answer in terms of π.

b. Find the radius of the cylinder.