9.5 Solve Quadratic Equations by Completing the Square

Before You solved quadratic equations by finding square roots.

Now You will solve quadratic equations by completing the square.

Why? So you can solve a problem about snowboarding, as in Ex. 50.

Key Vocabulary
- completing the square
- perfect square trinomial

COMMON CORE

CC.9-12.A.REI.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex soultions, and write them as $a \pm bi$ for real numbers a and b.

For an expression of the form $x^2 + bx$, you can add a constant c to the expression so that the expression $x^2 + bx + c$ is a perfect square trinomial. This process is called **completing the square**.

KEY CONCEPT *For Your Notebook*

Completing the Square

Words To complete the square for the expression $x^2 + bx$, add the square of half the coefficient of the term bx.

Algebra $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

EXAMPLE 1 Complete the square

Find the value of c that makes the expression $x^2 + 5x + c$ a perfect square trinomial. Then write the expression as the square of a binomial.

STEP 1 **Find** the value of c. For the expression to be a perfect square trinomial, c needs to be the square of half the coefficient of bx.

$c = \left(\frac{5}{2}\right)^2 = \frac{25}{4}$ Find the square of half the coefficient of bx.

STEP 2 **Write** the expression as a perfect square trinomial. Then write the expression as the square of a binomial.

$x^2 + 5x + c = x^2 + 5x + \frac{25}{4}$ Substitute $\frac{25}{4}$ for c.

$= \left(x + \frac{5}{2}\right)^2$ Square of a binomial

✓ GUIDED PRACTICE for Example 1

Find the value of c that makes the expression a perfect square trinomial. Then write the expression as the square of a binomial.

1. $x^2 + 8x + c$
2. $x^2 - 12x + c$
3. $x^2 + 3x + c$

SOLVING EQUATIONS The method of completing the square can be used to solve any quadratic equation. To use completing the square to solve a quadratic equation, you must write the equation in the form $x^2 + bx = d$.

EXAMPLE 2 Solve a quadratic equation

Solve $x^2 - 16x = -15$ by completing the square.

Solution

$x^2 - 16x = -15$	Write original equation.
$x^2 - 16x + (-8)^2 = -15 + (-8)^2$	Add $\left(\frac{-16}{2}\right)^2$, or $(-8)^2$, to each side.
$(x - 8)^2 = -15 + (-8)^2$	Write left side as the square of a binomial.
$(x - 8)^2 = 49$	Simplify the right side.
$x - 8 = \pm 7$	Take square roots of each side.
$x = 8 \pm 7$	Add 8 to each side.

▶ The solutions of the equation are $8 + 7 = 15$ and $8 - 7 = 1$.

CHECK You can check the solutions in the original equation.

If $x = 15$:	**If $x = 1$:**
$(15)^2 - 16(15) \stackrel{?}{=} -15$	$(1)^2 - 16(1) \stackrel{?}{=} -15$
$-15 = -15$ ✓	$-15 = -15$ ✓

AVOID ERRORS

When completing the square to solve an equation, be sure you add the term $\left(\frac{b}{2}\right)^2$ to both sides of the equation.

EXAMPLE 3 Solve a quadratic equation in standard form

Solve $2x^2 + 20x - 8 = 0$ by completing the square.

Solution

$2x^2 + 20x - 8 = 0$	Write original equation.
$2x^2 + 20x = 8$	Add 8 to each side.
$x^2 + 10x = 4$	Divide each side by 2.
$x^2 + 10x + 5^2 = 4 + 5^2$	Add $\left(\frac{10}{2}\right)^2$, or 5^2, to each side.
$(x + 5)^2 = 29$	Write left side as the square of a binomial.
$x + 5 = \pm\sqrt{29}$	Take square roots of each side.
$x = -5 \pm \sqrt{29}$	Subtract 5 from each side.

▶ The solutions are $-5 + \sqrt{29} \approx 0.39$ and $-5 - \sqrt{29} \approx -10.39$.

AVOID ERRORS

Be sure that the coefficient of x^2 is 1 before you complete the square.

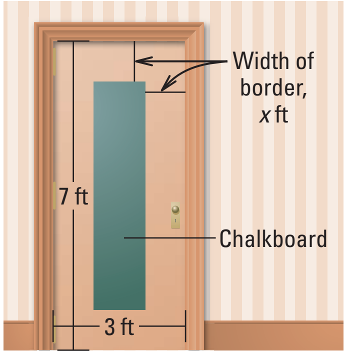
✓ GUIDED PRACTICE for Examples 2 and 3

Solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.

4. $x^2 - 2x = 3$ **5.** $m^2 + 10m = -8$ **6.** $3g^2 - 24g + 27 = 0$

EXAMPLE 4 Solve a multi-step problem

CRAFTS You decide to use chalkboard paint to create a chalkboard on a door. You want the chalkboard to have a uniform border as shown. You have enough chalkboard paint to cover 6 square feet. Find the width of the border to the nearest inch.



Solution

STEP 1 **Write** a verbal model. Then write an equation. Let x be the width (in feet) of the border.

Area of chalkboard (square feet)	=	Length of chalkboard (feet)	•	Width of chalkboard (feet)
6	$=$	$(7 - 2x)$	$\cdot$	$(3 - 2x)$

WRITE EQUATION
The width of the border is subtracted twice because it is at the top and the bottom of the door, as well as at the left and the right.

STEP 2 **Solve** the equation.

$$6 = (7 - 2x)(3 - 2x) \quad \text{Write equation.}$$

$$6 = 21 - 20x + 4x^2 \quad \text{Multiply binomials.}$$

$$-15 = 4x^2 - 20x \quad \text{Subtract 21 from each side.}$$

$$-\frac{15}{4} = x^2 - 5x \quad \text{Divide each side by 4.}$$

$$-\frac{15}{4} + \frac{25}{4} = x^2 - 5x + \frac{25}{4} \quad \text{Add } \left(-\frac{5}{2}\right)^2\text{, or } \frac{25}{4}\text{, to each side.}$$

$$-\frac{15}{4} + \frac{25}{4} = \left(x - \frac{5}{2}\right)^2 \quad \text{Write right side as the square of a binomial.}$$

$$\frac{5}{2} = \left(x - \frac{5}{2}\right)^2 \quad \text{Simplify left side.}$$

$$\pm\sqrt{\frac{5}{2}} = x - \frac{5}{2} \quad \text{Take square roots of each side.}$$

$$\frac{5}{2} \pm \sqrt{\frac{5}{2}} = x \quad \text{Add } \frac{5}{2} \text{ to each side.}$$

The solutions of the equation are $\frac{5}{2} + \sqrt{\frac{5}{2}} \approx 4.08$ and $\frac{5}{2} - \sqrt{\frac{5}{2}} \approx 0.92$.

It is not possible for the width of the border to be 4.08 feet because the width of the door is 3 feet. So, the width of the border is 0.92 foot. Convert 0.92 foot to inches.

$$0.92\ \cancel{\text{ft}} \cdot \frac{12 \text{ in.}}{1\ \cancel{\text{ft}}} = 11.04 \text{ in.} \quad \text{Multiply by conversion factor.}$$

▶ The width of the border should be about 11 inches.

✓ GUIDED PRACTICE for Example 4

7. **WHAT IF?** In Example 4, suppose you have enough chalkboard paint to cover 4 square feet. Find the width of the border to the nearest inch.

9.5 EXERCISES

HOMEWORK KEY

○ = See **WORKED-OUT SOLUTIONS** Exs. 19 and 47

★ = **STANDARDIZED TEST PRACTICE** Exs. 2, 24, 25, 42, and 49

◆ = **MULTIPLE REPRESENTATIONS** Ex. 47

SKILL PRACTICE

1. **VOCABULARY** Copy and complete: The process of writing an expression of the form $x^2 + bx$ as a perfect square trinomial is called _?_.

2. ★ **WRITING** Give an example of an expression that is a perfect square trinomial. *Explain* why the expression is a perfect square trinomial.

EXAMPLE 1 for Exs. 3–11

COMPLETING THE SQUARE **Find the value of *c* that makes the expression a perfect square trinomial. Then write the expression as the square of a binomial.**

3. $x^2 + 6x + c$
4. $x^2 + 12x + c$
5. $x^2 - 4x + c$
6. $x^2 - 8x + c$
7. $x^2 - 3x + c$
8. $x^2 + 5x + c$
9. $x^2 + 2.4x + c$
10. $x^2 - \frac{1}{2}x + c$
11. $x^2 - \frac{4}{3}x + c$

EXAMPLES 2 and 3 for Exs. 12–27

SOLVING EQUATIONS **Solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.**

12. $x^2 + 2x = 3$
13. $x^2 + 10x = 24$
14. $c^2 - 14c = 15$
15. $n^2 - 6n = 72$
16. $a^2 - 8a + 15 = 0$
17. $y^2 + 4y - 21 = 0$
18. $w^2 - 5w = \frac{11}{4}$
19. $z^2 + 11z = -\frac{21}{4}$
20. $g^2 - \frac{2}{3}g = 7$
21. $k^2 - 8k - 7 = 0$
22. $v^2 - 7v + 1 = 0$
23. $m^2 + 3m + \frac{5}{4} = 0$

24. ★ **MULTIPLE CHOICE** What are the solutions of $4x^2 + 16x = 9$?

Ⓐ $-\frac{1}{2}, -\frac{9}{2}$ Ⓑ $-\frac{1}{2}, \frac{9}{2}$ Ⓒ $\frac{1}{2}, -\frac{9}{2}$ Ⓓ $\frac{1}{2}, \frac{9}{2}$

25. ★ **MULTIPLE CHOICE** What are the solutions of $x^2 + 12x + 10 = 0$?

Ⓐ $-6 \pm \sqrt{46}$ Ⓑ $-6 \pm \sqrt{26}$ Ⓒ $6 \pm \sqrt{26}$ Ⓓ $6 \pm \sqrt{46}$

ERROR ANALYSIS ***Describe*** **and correct the error in solving the given equation.**

26. $x^2 - 14x = 11$

$$x^2 - 14x = 11$$
$$x^2 - 14x + 49 = 11$$
$$(x - 7)^2 = 11$$
$$x - 7 = \pm\sqrt{11}$$
$$x = 7 \pm \sqrt{11}$$

27. $x^2 - 2x - 4 = 0$

$$x^2 - 2x - 4 = 0$$
$$x^2 - 2x = 4$$
$$x^2 - 2x + 1 = 4 + 1$$
$$(x + 1)^2 = 5$$
$$x + 1 = \pm\sqrt{5}$$
$$x = 1 \pm \sqrt{5}$$

SOLVING EQUATIONS **Solve the equation by completing the square. Round your solutions to the nearest hundredth, if necessary.**

28. $2x^2 - 8x - 14 = 0$

29. $2x^2 + 24x + 10 = 0$

30. $3x^2 - 48x + 39 = 0$

31. $4y^2 + 4y - 7 = 0$

32. $9n^2 + 36n + 11 = 0$

33. $3w^2 - 18w - 20 = 0$

34. $3p^2 - 30p - 11 = 6p$

35. $3a^2 - 12a + 3 = -a^2 - 4$

36. $15c^2 - 51c - 30 = 9c + 15$

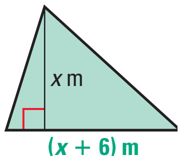
37. $7m^2 + 24m - 2 = m^2 - 9$

38. $g^2 + 2g + 0.4 = 0.9g^2 + g$

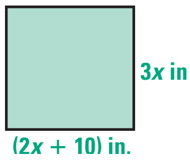
39. $11z^2 - 10z - 3 = -9z^2 + \frac{3}{4}$

GEOMETRY **Find the value of x. Round your answer to the nearest hundredth, if necessary.**

40. Area of triangle $= 108\text{ m}^2$



41. Area of rectangle $= 288\text{ in.}^2$



42. ★ **WRITING** How many solutions does $x^2 + bx = c$ have if $c < -\left(\frac{b}{2}\right)^2$? *Explain.*

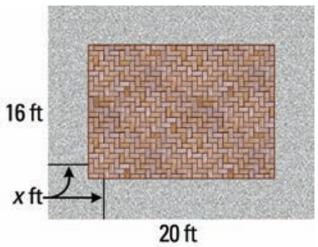
43. **CHALLENGE** The product of two consecutive negative integers is 210. Find the integers.

44. **CHALLENGE** The product of two consecutive positive even integers is 288. Find the integers.

PROBLEM SOLVING

EXAMPLE 4
for Exs. 45–46

45. **LANDSCAPING** You are building a rectangular brick patio surrounded by crushed stone in a rectangular courtyard as shown. The crushed stone border has a uniform width x (in feet). You have enough money in your budget to purchase patio bricks to cover 140 square feet. Solve the equation $140 = (20 - 2x)(16 - 2x)$ to find the width of the border.



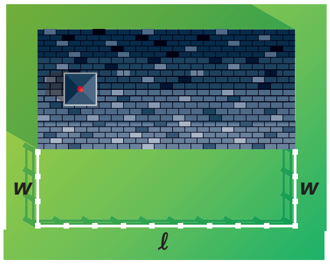
46. **TRAFFIC ENGINEERING** The distance d (in feet) that it takes a car to come to a complete stop on dry asphalt can be modeled by $d = 0.05s^2 + 1.1s$ where s is the speed of the car (in miles per hour). A car has 78 feet to come to a complete stop. Find the maximum speed at which the car can travel.

47. **MULTIPLE REPRESENTATIONS** For the period 1985–2001, the average salary y (in thousands of dollars) per season of a Major League Baseball player can be modeled by $y = 7x^2 - 4x + 392$ where x is the number of years since 1985.

a. **Solving an Equation** Write and solve an equation to find the year when the average salary was \$1,904,000.

b. **Drawing a Graph** Use a graph to check your solution to part (a).

48. **MULTI-STEP PROBLEM** You have 80 feet of fencing to make a rectangular horse pasture that covers 750 square feet. A barn will be used as one side of the pasture as shown.

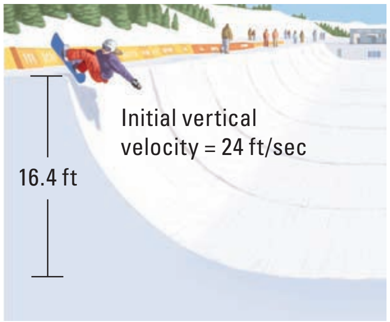


a. Write equations for the perimeter and area of the pasture.

b. Use substitution to solve the system of equations from part (a). What are the possible dimensions of the pasture?

49. ★ **SHORT RESPONSE** You purchase stock for \$16 per share, and you sell the stock 30 days later for \$23.50 per share. The price y (in dollars) of a share during the 30 day period can be modeled by $y = -0.025x^2 + x + 16$ where x is the number of days after the stock is purchased. Could you have sold the stock earlier for \$23.50 per share? *Explain.*

50. **SNOWBOARDING** During a "big air" competition, snowboarders launch themselves from a half pipe, perform tricks in the air, and land back in the half pipe.



Cross section of a half pipe

a. **Model** Use the vertical motion model to write an equation that models the height h (in feet) of a snowboarder as a function of the time t (in seconds) she is in the air.

b. **Apply** How long is the snowboarder in the air if she lands 13.2 feet above the base of the half pipe? Round your answer to the nearest tenth of a second.

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51. **CHALLENGE** You are knitting a rectangular scarf. The pattern you have created will result in a scarf that has a length of 60 inches and a width of 4 inches. However, you happen to have enough yarn to cover an area of 480 square inches. You decide to increase the dimensions of the scarf so that all of your yarn will be used. If the increase in the length is 10 times the increase in the width, what will the dimensions of the scarf be?