9.7 Solve Systems with Quadratic Equations

You solved systems of linear equations.

Now you will solve systems that include a quadratic equation.

Why?
So you can predict the path of a ball, as in Example 4.

You have solved systems of linear equations using the graph-and-check method and using the substitution method. You can use both of these techniques to solve a system of equations involving nonlinear equations, such as quadratic equations.

Recall that the substitution method consists of the following three steps.

**STEP 1** Solve one of the equations for one of its variables.

**STEP 2** Substitute the expression from Step 1 into the other equation and solve for the other variable.

**STEP 3** Substitute the value from Step 2 into one of the original equations and solve.

**EXAMPLE 1** Use the substitution method

**Solve the system:**

\[
egin{align*}
y &= 3x + 2 \quad \text{Equation 1} \\
y &= 3x^2 + 6x + 2 \quad \text{Equation 2}
\end{align*}
\]

**Solution**

**STEP 1** Solve one of the equations for \( y \). Equation 1 is already solved for \( y \).

**STEP 2** Substitute \( 3x + 2 \) for \( y \) in Equation 2 and solve for \( x \).

\[
\begin{align*}
y &= 3x^2 + 6x + 2 \\
3x + 2 &= 3x^2 + 6x + 2 \\
0 &= 3x^2 + 3x \\
0 &= 3x(x + 1) \\
3x &= 0 \quad \text{or} \quad x + 1 = 0 \\
x &= 0 \quad \text{or} \quad x = -1
\end{align*}
\]

Zero-product property

Solve for \( x \).

**STEP 3** Substitute both 0 and \(-1\) for \( x \) in Equation 1 and solve for \( y \).

\[
\begin{align*}
y &= 3x + 2 \\
y &= 3(0) + 2 \\
y &= 3(-1) + 2 \\
y &= 2 \quad \text{or} \quad y = -1
\end{align*}
\]

The solutions are (0, 2) and \((-1, -1)\).
POINTS OF INTERSECTION When you graph a system of equations, the graphs intersect at each solution of the system. For a system consisting of a linear equation and a quadratic equation the number of intersections, and therefore solutions, can be zero, one, or two.

KEY CONCEPT

For Your Notebook

Systems With One Linear Equation and One Quadratic Equation

There are three possibilities for the number of points of intersection.

No Solution One Solution Two Solutions

EXAMPLE 2 Use a graphing calculator to solve a system

Solve the system:

\[ y = 2x - 4 \]  \hspace{1cm} \text{Equation 1}
\[ y = x^2 - 4x + 1 \]  \hspace{1cm} \text{Equation 2}

Solution

\text{STEP 1} \quad \text{Enter each equation into your graphing calculator.}
Set \( Y_1 = 2x - 4 \) and \( Y_2 = x^2 - 4x + 1 \).

\text{STEP 2} \quad \text{Graph the system. Set a good viewing window. For this system, a good window is } -10 \leq x \leq 10 \text{ and } -10 \leq y \leq 10.

\text{STEP 3} \quad \text{Use the Trace function to find the coordinates of each point of intersection. The points of intersection are } (1, -2) \text{ and } (5, 6).

\[ \text{Intersection } X=1 \quad Y=-2 \]
\[ \text{Intersection } X=5 \quad Y=6 \]

\( \checkmark \) The solutions are \( (1, -2) \) and \( (5, 6) \).

CHECK Check the solutions. For example, check \( (1, -2) \).
\[ y = 2x - 4 \]
\[ -2 = 2(1) - 4 \]
\[ -2 = -2 \checkmark \]
\[ y = x^2 - 4x + 1 \]
\[ -2 = (1)^2 - 4(1) + 1 \]
\[ -2 = -2 \checkmark \]
GUIDED PRACTICE for Examples 1 and 2

Solve the system of equations first by using the substitution method and then by using a graphing calculator.

1. \( y = x + 4 \)
   \[ y = 2x^2 - 3x - 2 \]

2. \( y = x + 1 \)
   \[ y = -x^2 + 6x + 1 \]

3. \( y = x^2 - 6x + 11 \)
   \[ y = x + 1 \]

SOLVING EQUATIONS
You can use a graph to solve an equation in one variable. Treat each side of the equation as a function. Then graph each function on the same coordinate plane. The x-value of any points of intersection will be the solutions of the equation.

EXAMPLE 3 Solve an equation using a system

Solve the equation \(-x^2 - 4x + 2 = -2x - 1\) using a system of equations. Check your solution(s).

Solution

**STEP 1** Write a system of two equations by setting both the left and right sides of the given equation each equal to \( y \).

\[-x^2 - 4x + 2 = -2x - 1\]

\[ y = -x^2 - 4x + 2 \] \hspace{1cm} \text{Equation 1}

\[ y = -2x - 1 \] \hspace{1cm} \text{Equation 2}

**STEP 2** Graph Equation 1 and Equation 2 on the same coordinate plane or on a graphing calculator.

**STEP 3** The x-value of each point of intersection is a solution of the original equation. The graphs intersect at \((-3, 5)\) and \((1, -3)\).

The solutions of the equation are \( x = -3 \) and \( x = 1 \).

**CHECK:** Substitute each solution in the original equation.

\[-x^2 - 4x + 2 = -2x - 1\]

\[-(-3)^2 - 4(-3) + 2 \neq -2(-3) - 1\]

\[ -9 + 12 + 2 \neq 6 - 1\]

\[ 5 = 5 \checkmark \]

\[-3 = -3 \checkmark \]

GUIDED PRACTICE for Example 3

Solve the equation using a system of equations.

4. \( x + 3 = 2x^2 + 3x - 1 \)

5. \( x^2 + 7x + 4 = 2x + 4 \)

6. \( 8 = x^2 - 4x + 3 \)

7. \( -x + 4 = 3^5 \)
EXAMPLE 4  Solve a multi-step problem

BASEBALL  During practice, you hit a baseball toward the gym, which is 240 feet away.
The path of the baseball after it is hit can be modeled by the equation:
\[ y = -0.004x^2 + x + 3 \]
The roof of the gym can be modeled by the equation:
\[ y = \frac{2}{3}x - 120 \]
for values of \( x \) greater than 240 feet and less than 320 feet.
The wall of the gym can be modeled by the equation:
\[ x = 240 \]
for values of \( y \) between 0 feet and 40 feet.
Does the baseball hit the roof of gym?

Solution

**STEP 1**  Write a system of two equations for the baseball and the roof.
\[ y = -0.004x^2 + x + 3 \quad \text{Equation 1 (baseball)} \]
\[ y = \frac{2}{3}x - 120 \quad \text{Equation 2 (roof)} \]

**STEP 2**  Graph both equations on the same coordinate plane.

**STEP 3**  The \( x \)-value where the graphs intersect is between 200 feet and 230 feet which is outside the domain of the equation for the roof.

\[ \checkmark \] The baseball does not hit the roof.

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**Guided Practice for Example 4**

8. **WHAT IF?** In Example 4, does the baseball hit the gym wall? If it does, how far up the wall does it hit? If it does not, how far away from the gym wall does the ball land?

9. **WHAT IF?** In Example 4, if you hit the ball so that it followed a path that had a smaller number as the coefficient of \( x^2 \), would it be more or less likely to hit the gym? *Explain.*
1. **VOCABULARY** Describe how to use the substitution method to solve a system of linear equations.

2. **WRITING** Describe the possible number of solutions for a system consisting of a linear equation and a quadratic equation.

**SUBSTITUTION METHOD** Solve the system of equations using the substitution method.

3. \( y = x^2 - x + 2 \)
4. \( y = -x^2 + 4x - 2 \)
5. \( y = x^2 - x \)
   
   \( y = x + 5 \)
   \( y = 4x - 6 \)
   \( y = -\frac{5}{2} x + 1 \)

6. \( y = 2x^2 + x - 1 \)
7. \( y = 3x^2 - 6 \)
8. \( y = -2x^2 - 2x + 3 \)
   
   \( y = -x - 3 \)
   \( y = -3x \)
   \( y = \frac{7}{2} \)

9. **ERROR ANALYSIS** Describe and correct the error in the solution steps shown.

\[
\begin{align*}
    y &= 3x^2 - 6x + 4 & \text{Equation 1} \\
    y &= 4 & \text{Equation 2} \\
    y &= 3(4)^2 - 6(4) + 4 & \text{Substitute.} \\
    y &= 3(16) - 24 + 4 = 28 & \text{Simplify}
\end{align*}
\]

10. **COPY AND COMPLETE** When a system of equations includes a linear equation and a quadratic equation, there will (*always, sometimes, never*) be an infinite number of solutions.

11. **MULTIPLE CHOICE** Which equation intersects the graph of \( y = x^2 - 4x + 3 \) twice?

    - \( A \) \( y = -1 \)
    - \( B \) \( x = 2 \)
    - \( C \) \( y + 1 = x \)
    - \( D \) \( y + x = -1 \)

**GRAPHING CALCULATOR** Use a graphing calculator to find the points of intersection, if any, of the graph of the system of equations.

12. \( y = 3x^2 - 2x + 1 \)
    \( y = x + 7 \)
13. \( y = x^2 + 2x + 11 \)
    \( y = -2x + 8 \)
14. \( y = -2x^2 - 4x \)
    \( y = 2 \)
15. \( y = \frac{1}{2} x^2 - 3x + 4 \)
    \( y = x - 2 \)
16. \( y = \frac{1}{3} x^2 + 2x - 3 \)
    \( y = 2x \)
17. \( y = 4x^2 + 5x - 7 \)
    \( y = -3x + 5 \)

**SOLVE THE EQUATION** Solve the equation using a system. Check each answer.

18. \(-5x + 5 = x^2 - 4x + 3\)
19. \(-6 = x^2 + 2x - 5\)
20. \(-3 = x^2 + 5x - 3\)
21. \(2x^2 + 4x = 2x + 4\)
GRAPHING CALCULATOR Use a graphing calculator to find the points of intersection, if any, of the graph of the system of equations.

22. \( y = -x^2 + 4 \)
23. \( y = -1 \)
24. \( y = x + 6 \)
25. \( y = -x^2 + 2x \)
26. \( y = 3x - 1 \)
27. \( y = -1.5x + 1 \)
28. ★ WRITING Describe the possible number of solutions for a system consisting of a quadratic equation and an exponential equation.

SOLVE THE EQUATION Solve the equation using a system.

29. \( 2^x + 1 = 2x + 1 \)
30. \( 4^x = -\frac{2}{3}x + 9 \)
31. \( -2x + 11 = 2^x - 3 \)
32. \( 3^x - 5 = 6x - 8 \)
33. CHALLENGE Using a graphing calculator, find the points of intersection, if any, of the graphs of the equations \( y = x^2 - 3x + 1 \) and \( y = x^2 - x - 1 \). What are the solutions of the system?

**Problem Solving**

34. RECREATION Marion and Reggie are driving boats on the same lake. Marion’s chosen path can be modeled by the equation \( y = -x^2 - 4x - 1 \) and Reggie’s path can be modeled by the equation \( y = 2x + 8 \). Do their paths cross each other? If so, what are the coordinates of the point(s) where the paths meet?

35. ★ SHORT RESPONSE Two dogs are running in a fenced dog park. One dog is following a path that can be modeled by the equation \( y = 4 \). Another dog is following a path that can be modeled by the equation \( y = -x^2 + 3 \). Will the dogs’ paths cross? Explain your answer.

36. ARCHITECTURE The arch of the Sydney Harbor Bridge in Sydney, Australia, can be modeled by \( y = -0.00211x^2 + 1.06x \) where \( x \) is the distance (in meters) from the left pylons and \( y \) is the height (in meters) of the arch above the water. The road can be modeled by the equation \( y = 52 \). To the nearest meter, how far from the left pylons are the two points where the road intersects the arch of the bridge?
37. **SAVINGS** Nancy and Miranda are looking at different ways to save money. Graph the two equations. Explain what happens when the graphs intersect. When will Miranda have more money saved than Nancy?

**Nancy**
I will save $15 each month.
My money will not earn interest.
A model for my savings is \( y = 15x \).
\( x \) is the number of months
\( y \) is my total savings

**Miranda**
I will put $200 into an account that earns 2% annually. I will not save any more money.
If the interest is compounded monthly, \( y = 200(1.02)^x \) models my savings.
\( x \) is months and \( y \) is my total savings.

38. **SPACE** Suppose an asteroid and a piece of space debris are traveling in the same plane in space. The asteroid follows a path that can be modeled locally by the equation \( y = 2x^2 - 3x + 1 \). The space debris follows a path that can be modeled locally by the equation \( y = 8x - 13 \).

a. Will the paths of the two objects intersect? Is it possible for the two objects to collide? If so, what are the coordinates of the point where the paths intersect?

b. What additional information would you need to decide whether the two objects will collide? Explain.

39. **MULTI-STEP PROBLEM** Keno asks Miguel if the graphs of all three of the equations shown below ever intersect in a single point.

\[
\begin{align*}
y &= 3x + 1 \\
y &= 2x^2 - 4x + 6 \\
y &= -2x + 6 
\end{align*}
\]

a. Find any points of intersection of the graphs of \( y = 3x + 1 \) and \( y = 2x^2 - 4x + 6 \).

b. Find any points of intersection of the graphs of \( y = 3x + 1 \) and \( y = -2x + 6 \).

c. Find any points of intersection of the graphs \( y = 2x^2 - 4x + 6 \) and \( y = -2x + 6 \).

d. Do the three graphs ever intersect in a single point? If so, what are the coordinates of this point of intersection?

40. **CHALLENGE** Find the point(s) of intersection, if any, for the line with equation \( y = -x - 1 \) and the circle with equation \( x^2 + y^2 = 41 \).